optimization and approximation

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Európska únia
Európsky sociálny fond
Operačný program VZDELÁVANIE
VÝSKUMNÁ AGENTÚRA
Dopytovo orientovaný projekt
Moderné vzdelávanie pre vedomostnú spoločnosť
Projekt je spolufinancovaný zo zdrojov EÚ

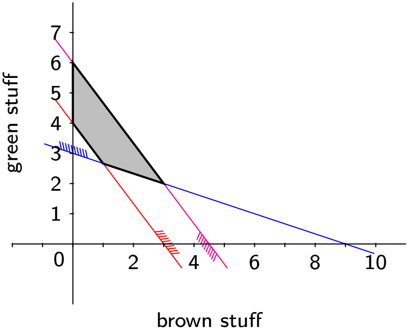
linear programming

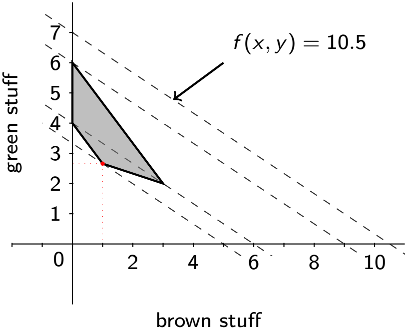
dl of stuff

	brown		green			
minimize	x	$+$	$1.5y$	$=:$	$f(x,y)$	*price*
subject to	$30x$	$+$	$90y$	$\geq$	270	*caffeine*
	$40x$	$+$	$30y$	$\geq$	120	*sugar*
	$40x$	$+$	$30y$	$\leq$	180	*aspartame*
			x, y	$\geq$	0	

linear programming

$$
\begin{array}{lcccccll}
 & \multicolumn{3}{c}{\text{dl of stuff}} & & & & \\
 & \text{brown} & & \text{green} & & & & \\
\text{minimize} & x & + & 1.5y & =: & f(x,y) & & \textit{price} \\
\text{subject to} & 30x & + & 90y & \geq & 270 & & \textit{caffeine} \\
 & 40x & + & 30y & \geq & 120 & & \textit{sugar} \\
 & 40x & + & 30y & \leq & 180 & & \textit{aspartame} \\
 & & & x, y & \geq & 0 & &
\end{array}
$$





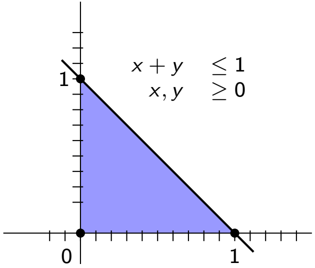
linear programming

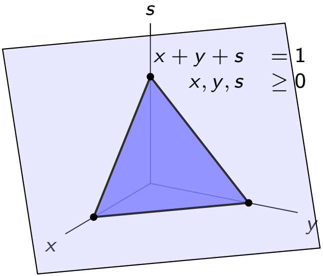
$$
\begin{array}{rccccccl}
 & \multicolumn{3}{c}{\text{dl of stuff}} & & & & \\
 & \text{brown} & & \text{green} & & & & \\
\text{minimize} & x & + & 1.5y & =: & f(x,y) & & \textit{price} \\
\text{subject to} & 30x & + & 90y & \geq & 270 & & \textit{caffeine} \\
 & 40x & + & 30y & \geq & 120 & & \textit{sugar} \\
 & 40x & + & 30y & \leq & 180 & & \textit{aspartame} \\
 & & & x, y & \geq & 0 & &
\end{array}
$$

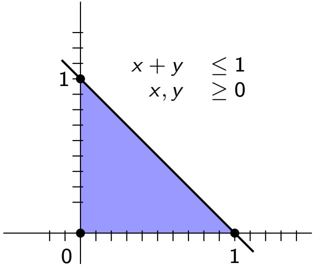
$$
\begin{array}{rrrcccll}
\text{maximmize} & -x & -1.5y & & & & =: & f(x,y,s_1,s_2,s_3) \\
\text{subject to} & -30x & -90y & +s_1 & & & = & -270 \\
 & -40x & -30y & & +s_2 & & = & -120 \\
 & 40x & +30y & & & +s_3 & = & 180 \\
 & & & \multicolumn{3}{r}{x, y, s_1, s_2, s_3} & \geq & 0
\end{array}
$$

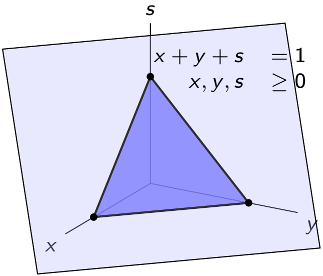
$$
\mathbf{c} = \begin{pmatrix} -1 \\ -1.5 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\qquad
A = \begin{pmatrix} -30 & -90 & 1 & 0 & 0 \\ -40 & -30 & 0 & 1 & 0 \\ 30 & 40 & 0 & 0 & 1 \end{pmatrix}
\qquad
\mathbf{b} = \begin{pmatrix} -270 \\ -120 \\ 200 \end{pmatrix}
$$

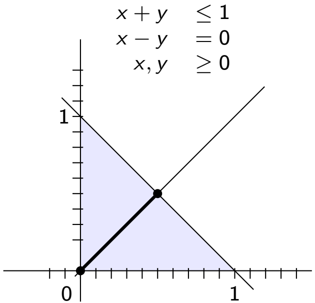
$$
\max_{\mathbf{x} \in \mathbb{R}^5} \left\{ \mathbf{c}^{\mathsf{T}} \mathbf{x} \mid A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0 \right\}.
$$

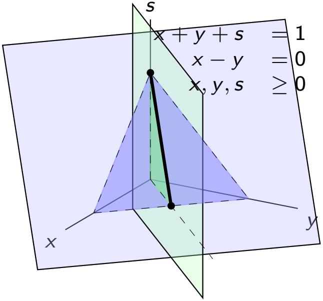

$x + y \leq 1$
$x, y \geq 0$
1
0
1


s
$x + y + s = 1$
$x, y, s \geq 0$
x
y


$x + y \leq 1$
$x, y \geq 0$
1
0
1


s
$x + y + s = 1$
$x, y, s \geq 0$
x
y

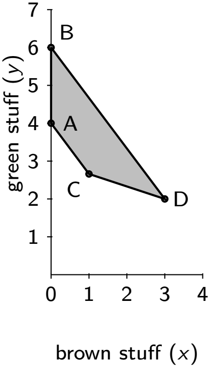

$x + y \leq 1$
$x - y = 0$
$x, y \geq 0$
1
0
1


s
$x + y + s = 1$
$x - y = 0$
$x, y, s \geq 0$
x
y

$$\max_{\mathbf{x}\in\mathbb{R}^n} \left\{\mathbf{c}^\mathsf{T}\mathbf{x} \mid A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0\right\}$$

$$\mathbf{c} = \begin{pmatrix} -1 \\ -1.5 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad A = \begin{pmatrix} -30 & -90 & 1 & 0 & 0 \\ -40 & -30 & 0 & 1 & 0 \\ 30 & 40 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} -270 \\ -120 \\ 200 \end{pmatrix}$$

constraints	x	y	s_1	s_2	s_3	
$x = y = 0$	0	0	−270	−120	180	
$x = s_1 = 0$	0	3	0	−30	90	
$x = s_2 = 0$	0	4	90	0	60	A
$x = s_3 = 0$	0	6	270	60	0	B
$y = s_1 = 0$	9	0	0	240	−180	
$y = s_2 = 0$	3	0	−180	0	60	
$y = s_3 = 0$	4.5	0	−135	60	0	
$\mathbf{s_1 = s_2 = 0}$	**1**	$\mathbf{\frac{8}{3}}$	**0**	**0**	**60**	C
$s_1 = s_3 = 0$	3	2	0	60	0	D
$s_2 = s_3 = 0$	no solution					



$$A = \begin{pmatrix} -30 & -90 & 1 & 0 & 0 \\ -40 & -30 & 0 & 1 & 0 \\ 30 & 40 & 0 & 0 & 1 \end{pmatrix} \quad A_{\{1,2\}} = \begin{pmatrix} -30 & -90 \\ -40 & -30 \\ 30 & 40 \end{pmatrix} \quad A_{\{3,4,5\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\max_{\mathbf{x} \in \mathbb{R}^n} \left\{ \mathbf{c}^\mathsf{T}\mathbf{x} \mid A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0 \right\}$, $A \in \mathbb{R}^{m \times n}$

basic solution: $\mathbf{x} \in \mathbb{R}^n$, there is a set $B \subseteq \{1, \ldots, n\}$, $|B| = m$:

1. the matrix $A_B \in \mathbb{R}^{m \times m}$ has full rank, m (i.e. is non-singular)
2. $x_j = 0$ for all $j \notin B$

$$A = \begin{pmatrix} -30 & -90 & 1 & 0 & 0 \\ -40 & -30 & 0 & 1 & 0 \\ 30 & 40 & 0 & 0 & 1 \end{pmatrix} \quad A_{\{1,2\}} = \begin{pmatrix} -30 & -90 \\ -40 & -30 \\ 30 & 40 \end{pmatrix} \quad A_{\{3,4,5\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\max_{\mathbf{x} \in \mathbb{R}^n} \left\{ \mathbf{c}^\mathsf{T}\mathbf{x} \mid A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0 \right\}$, $A \in \mathbb{R}^{m \times n}$

basic solution: $\mathbf{x} \in \mathbb{R}^n$, there is a set $B \subseteq \{1, \ldots, n\}$, $|B| = m$:

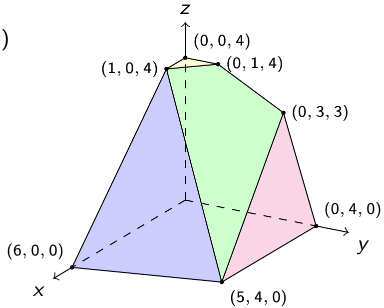
1. the matrix $A_B \in \mathbb{R}^{m \times m}$ has full rank, m (i.e. is non-singular)
2. $x_j = 0$ for all $j \notin B$

Let $\mathbf{c}^\mathsf{T}\mathbf{x}$ be bounded from above. Let $\mathbf{x_0}$ be feasible.

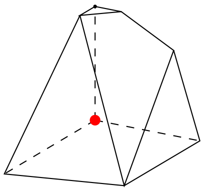
there is some **feasible basic solution** $\tilde{\mathbf{x}}$, for which $\mathbf{c}^\mathsf{T}\tilde{\mathbf{x}} \geq \mathbf{c}^\mathsf{T}\mathbf{x_0}$.

simplex method

$$
\begin{array}{lllll}
\text{maximize} & x & +y & +z & =: f(x,y,z) \\
\text{subject to} & x & +y & +2z & \le 9 \\
 & 4x & +y & +5z & \le 24 \\
 & & 3y & +z & \le 12 \\
 & & & z & \le 4 \\
 & & & x,y,z & \ge 0
\end{array}
$$

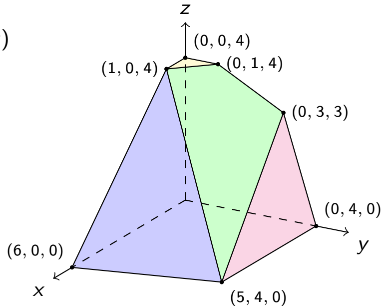


$$
\begin{array}{lllll}
f = & & x & +y & +z \\
\hline
s_1 = & 9 & -x & -y & -2z \\
s_2 = & 24 & -4x & -y & -5z \\
s_3 = & 12 & & -3y & -z \\
s_4 = & 4 & & & -z
\end{array}
$$

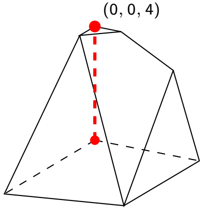


simplex method

$$\begin{array}{llllll}
\text{maximize} & x & +y & +z & =: f(x,y,z) \\
\text{subject to} & x & +y & +2z & \leq 9 \\
& 4x & +y & +5z & \leq 24 \\
& & 3y & +z & \leq 12 \\
& & & z & \leq 4 \\
& & & x,y,z & \geq 0
\end{array}$$

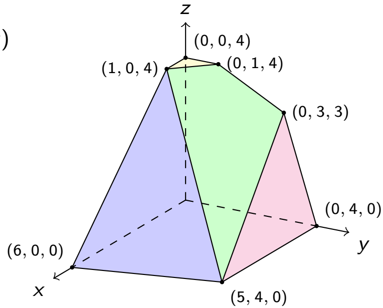


$$\begin{array}{lllll}
f = & 4 & +x & +y & -s_4 \\
\hline
z = & 4 & & & -s_4 \\
s_1 = & 1 & -x & -y & +2s_4 \\
s_2 = & 4 & -4x & -y & +5s_4 \\
s_3 = & 8 & & -3y & +s_4
\end{array}$$

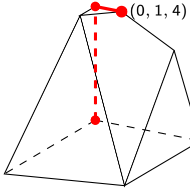


simplex method

$$\begin{aligned}
\text{maximize} \quad & x + y + z =: f(x,y,z) \\
\text{subject to} \quad & x + y + 2z \le 9 \\
& 4x + y + 5z \le 24 \\
& 3y + z \le 12 \\
& z \le 4 \\
& x, y, z \ge 0
\end{aligned}$$

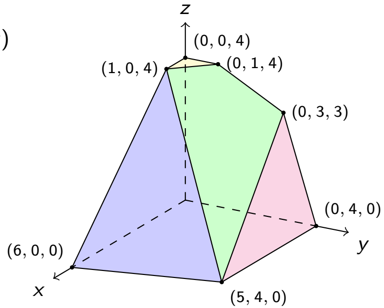


$$\begin{array}{rlllll}
f = & 5 & & - s_1 & + s_4 \\
\hline
y = & 1 & - x & - s_1 & + 2s_4 \\
z = & 4 & & & - s_4 \\
s_2 = & 3 & - 3x & + s_1 & + 3s_4 \\
s_3 = & 5 & + 3x & + 3s_1 & - 5s_4
\end{array}$$

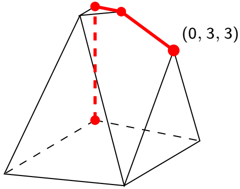


simplex method

$$
\begin{array}{llllll}
\text{maximize} & x & +y & +z & =: f(x,y,z) \\
\text{subject to} & x & +y & +2z & \leq 9 \\
 & 4x & +y & +5z & \leq 24 \\
 & & 3y & +z & \leq 12 \\
 & & & z & \leq 4 \\
 & & & x,y,z & \geq 0
\end{array}
$$

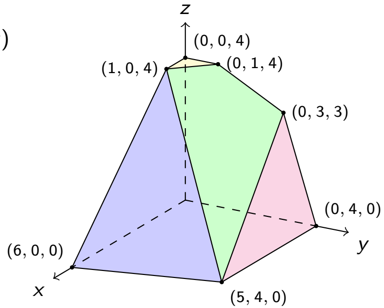


$$
\begin{array}{rcrrrr}
f & = & 6 & +\frac{3}{5}x & -\frac{2}{5}s_1 & -\frac{1}{5}s_3 \\
\hline
y & = & 3 & +\frac{1}{5}x & +\frac{1}{5}s_1 & -\frac{2}{5}s_3 \\
z & = & 3 & -\frac{3}{5}x & -\frac{3}{5}s_1 & +\frac{1}{5}s_3 \\
s_2 & = & 6 & -\frac{6}{5}x & +\frac{14}{5}s_1 & -\frac{3}{5}s_3 \\
s_4 & = & 1 & +\frac{3}{5}x & +\frac{3}{5}s_1 & -\frac{1}{5}s_3
\end{array}
$$

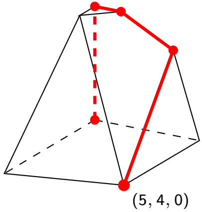


simplex method

$$
\begin{aligned}
\text{maximize} \quad & x + y + z =: f(x,y,z) \\
\text{subject to} \quad & x + y + 2z \le 9 \\
& 4x + y + 5z \le 24 \\
& 3y + z \le 12 \\
& z \le 4 \\
& x, y, z \ge 0
\end{aligned}
$$

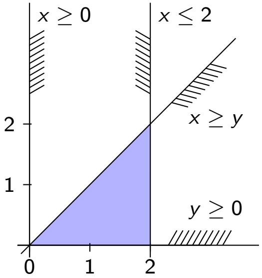


$$
\begin{aligned}
f &= 9 - z - s_1 \\
\hline
x &= 5 - \tfrac{5}{3}z - s_1 + \tfrac{1}{3}s_3 \\
y &= 4 - \tfrac{1}{3}z - \tfrac{1}{3}s_3 \\
s_2 &= 2z + 4s_1 - s_3 \\
s_4 &= 4 - z
\end{aligned}
$$



degenerate steps and infinite loops

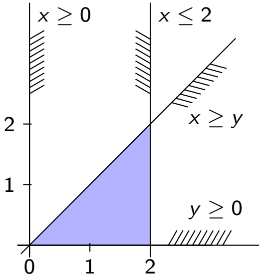
$$\begin{array}{lrrl} \text{maximize} & & y & =: f(x,y) \\ \text{subject to} & -x & +y & \leq 0 \\ & x & & \leq 2 \\ & & x, y & \geq 0 \end{array}$$



$$\begin{array}{rrrr} f = & & & y \\ \hline s_1 = & & x & -y \\ s_2 = & 2 & -x & \end{array}$$

degenerate steps and infinite loops

$$\begin{array}{lrrl} \text{maximize} & & y & =: f(x,y) \\ \text{subject to} & -x & +y & \leq 0 \\ & x & & \leq 2 \\ & & x,y & \geq 0 \end{array}$$

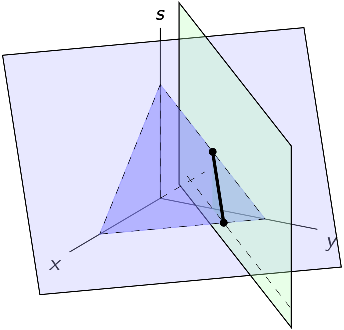


$$\begin{array}{rrrr} f = & & & y \\ \hline s_1 = & & x & -y \\ s_2 = & 2 & -x & \end{array}$$

$$\begin{array}{rrrr} f = & & x & -s_1 \\ \hline y = & & x & -s_1 \\ s_2 = & 2 & -x & \end{array}$$

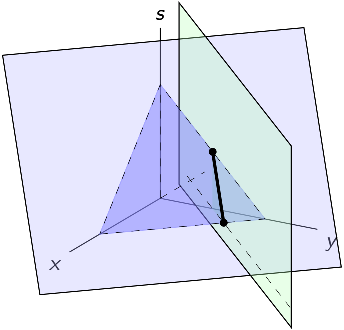
how to start

$$\begin{array}{rrrrl} \text{maximize} & 4x & & -z & =: f(x,y,z) \\ \text{subject to} & x & +y & +z & = 4 \\ & x & -y & & = -2 \\ & & & x,y,z & \geq 0 \end{array}$$



how to start

$$\begin{array}{rl} \text{maximize} & 4x - z =: f(x,y,z) \\ \text{subject to} & x + y + z = 4 \\ & x - y = -2 \\ & x, y, z \geq 0 \end{array}$$

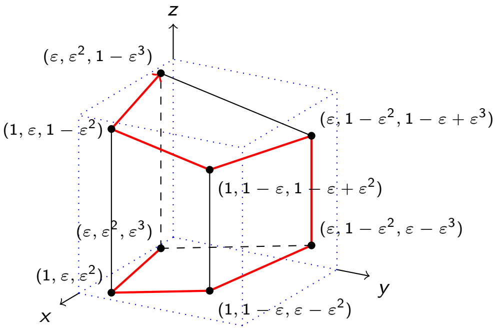


$$p_1 := 4 - x - y - z$$
$$p_2 := 2 + x - y$$

$$\begin{array}{rl} \text{maximize} & -p_1 - p_2 \\ \text{subject to} & p_1 + x + y + z = 4 \\ & p_2 - x + y = 2 \\ & x, y, z, p_1, p_2 \geq 0 \end{array}$$

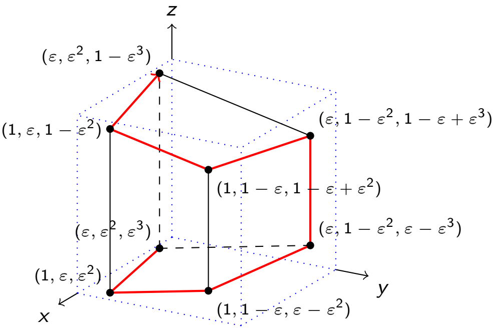
complexity

worst case exponential

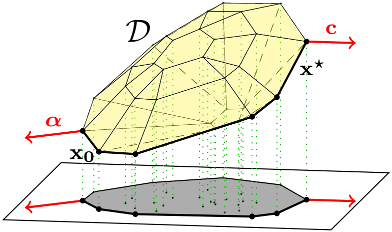


complexity

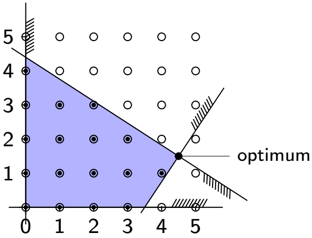
worst case exponential



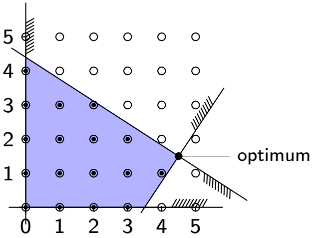
smoothed polynomial

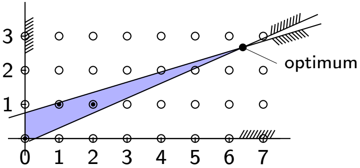


roundung is hard

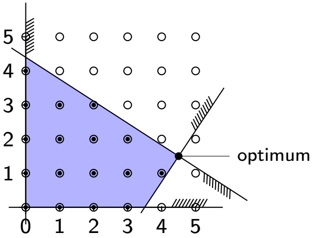


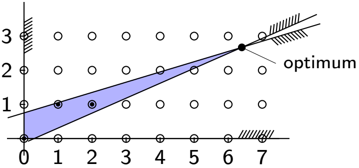
roundung is hard: as hard as SAT





roundung is hard

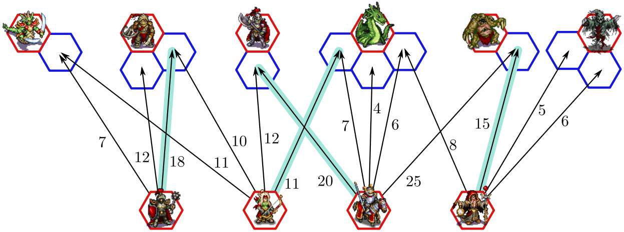




$$(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_3)$$

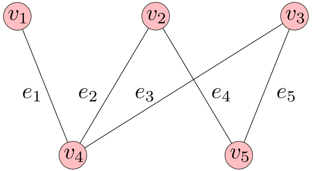
$$\begin{aligned}
x_1 + x_2 + x_3 &\geq 1 \\
(1 - x_1) + (1 - x_2) + (1 - x_3) &\geq 1 \\
x_1 + (1 - x_2) + (1 - x_3) &\geq 1 \\
(1 - x_1) + (1 - x_2) + x_3 &\geq 1 \\
(1 - x_1) + x_2 + x_3 &\geq 1 \\
x_1, x_2, x_3 &\geq 0 \\
x_1, x_2, x_3 &\leq 1
\end{aligned}$$

the good: Max-W-Bipartite-Matching



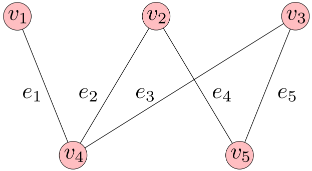
$$\begin{array}{llll}
\text{maximize} & \sum_{e \in E} \omega_e x_e & & \\
\text{subject to} & \sum_{\substack{e \in E \\ e=(v,w)}} x_e \leq 1 & & \forall v \in V \\
& x_e \geq 0 & & \forall e \in E \\
& x_e \in \mathbb{Z} & &
\end{array}$$

the good: Max-W-Bipartite-Matching



$$
\begin{aligned}
x_{e_1} &\leq 1 \\
x_{e_2} + x_{e_4} &\leq 1 \\
x_{e_3} + x_{e_5} &\leq 1 \\
x_{e_1} + x_{e_2} + x_{e_3} &\leq 1 \\
x_{e_4} + x_{e_5} &\leq 1 \\
x_{e_1}, x_{e_2}, x_{e_3}, x_{e_4}, x_{e_5} &\geq 0
\end{aligned}
$$

the good: Max-W-Bipartite-Matching

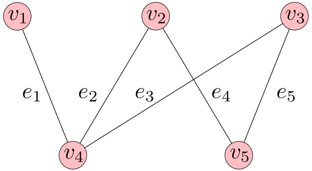


$$
\begin{aligned}
x_{e_1} &\le 1\\
x_{e_2} + x_{e_4} &\le 1\\
x_{e_3} + x_{e_5} &\le 1\\
x_{e_1} + x_{e_2} + x_{e_3} &\le 1\\
x_{e_4} + x_{e_5} &\le 1\\
x_{e_1}, x_{e_2}, x_{e_3}, x_{e_4}, x_{e_5} &\ge 0
\end{aligned}
$$

$$\max\{\mathbf{c}^{\mathsf{T}}\mathbf{x} \mid A\mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}\}$$

$$\mathbf{c} = \begin{pmatrix} 1 \\ 10 \\ 10 \\ 10 \\ 10 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x_{e_1} \\ x_{e_2} \\ x_{e_3} \\ x_{e_4} \\ x_{e_5} \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

the good: Max-W-Bipartite-Matching



$$
\begin{aligned}
x_{e_1} &\leq 1 \\
x_{e_2} + x_{e_4} &\leq 1 \\
x_{e_3} + x_{e_5} &\leq 1 \\
x_{e_1} + x_{e_2} + x_{e_3} &\leq 1 \\
x_{e_4} + x_{e_5} &\leq 1 \\
x_{e_1}, x_{e_2}, x_{e_3}, x_{e_4}, x_{e_5} &\geq 0
\end{aligned}
$$

$$\max\{\tilde{\mathbf{c}}^{\mathsf{T}}\tilde{\mathbf{x}} \mid \tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}},\ \tilde{\mathbf{x}} \geq \mathbf{0}\}$$

$$\tilde{\mathbf{c}} = \begin{pmatrix} 1 \\ 10 \\ 10 \\ 10 \\ 10 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \tilde{A} = \left(\begin{array}{ccccc|cccc} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \quad \tilde{\mathbf{x}} = \begin{pmatrix} x_{e_1} \\ x_{e_2} \\ x_{e_3} \\ x_{e_4} \\ x_{e_5} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} \quad \tilde{\mathbf{b}} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

the good: Max-W-Bipartite-Matching

$$\max\{\tilde{\mathbf{c}}^{\mathsf{T}}\tilde{\mathbf{x}} \mid \tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}},\ \tilde{\mathbf{x}} \geq \mathbf{0}\}$$

$$B = (2, 4, 7, 8)$$

$$\tilde{A} = \left(\begin{array}{ccccc|cccc} 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 1 & \mathbf{0} & \mathbf{0} & 0 \\ 0 & \mathbf{0} & 1 & \mathbf{0} & 1 & 0 & \mathbf{1} & \mathbf{0} & 0 \\ 1 & \mathbf{1} & 1 & \mathbf{0} & 0 & 0 & \mathbf{0} & \mathbf{1} & 0 \\ 0 & \mathbf{0} & 0 & \mathbf{1} & 1 & 0 & \mathbf{0} & \mathbf{0} & 1 \end{array}\right)$$

$$\tilde{A}_B\tilde{\mathbf{x}}_B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{e_2} \\ x_{e_4} \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Cramer rule

$$\frac{\det\left(\tilde{A}_B\langle i\rangle\right)}{\det\left(\tilde{A}_B\right)}$$

the good: Max-W-Bipartite-Matching

$$\max\{\tilde{\mathbf{c}}^{\mathsf{T}}\tilde{\mathbf{x}} \mid \tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}},\ \tilde{\mathbf{x}} \geq \mathbf{0}\}$$

$$B = (2, 4, 7, 8)$$

$$\tilde{A} = \left(\begin{array}{ccccc|cccc} 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 1 & \mathbf{0} & \mathbf{0} & 0 \\ 0 & \mathbf{0} & 1 & \mathbf{0} & 1 & 0 & \mathbf{1} & \mathbf{0} & 0 \\ 1 & \mathbf{1} & 1 & \mathbf{0} & 0 & 0 & \mathbf{0} & \mathbf{1} & 0 \\ 0 & \mathbf{0} & 0 & \mathbf{1} & 1 & 0 & \mathbf{0} & \mathbf{0} & 1 \end{array}\right)$$

$$\det\left(\tilde{A}_B\right) = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{vmatrix} = 1$$

$$\tilde{A}_B\tilde{\mathbf{x}}_B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{e_2} \\ x_{e_4} \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{vmatrix} \mathbf{1} & 1 & 0 & 0 \\ \mathbf{1} & 0 & 1 & 0 \\ \mathbf{1} & 0 & 0 & 1 \\ \mathbf{1} & 1 & 0 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 1 & 0 \\ 1 & \mathbf{1} & 0 & 1 \\ 0 & \mathbf{1} & 0 & 0 \end{vmatrix} = 1$$

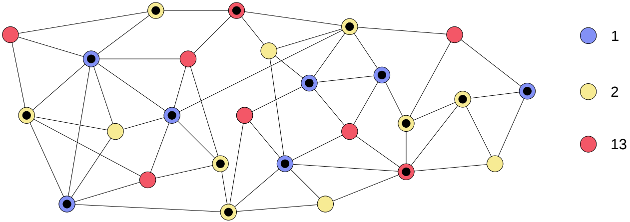
$$\begin{vmatrix} 1 & 1 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 1 & 0 & \mathbf{1} & 1 \\ 0 & 1 & \mathbf{1} & 0 \end{vmatrix} = 1 \qquad \begin{vmatrix} 1 & 1 & 0 & \mathbf{1} \\ 0 & 0 & 1 & \mathbf{1} \\ 1 & 0 & 0 & \mathbf{1} \\ 0 & 1 & 0 & \mathbf{1} \end{vmatrix} = 1$$

Cramer rule

$$\frac{\det\left(\tilde{A}_B\langle i\rangle\right)}{\det\left(\tilde{A}_B\right)}$$

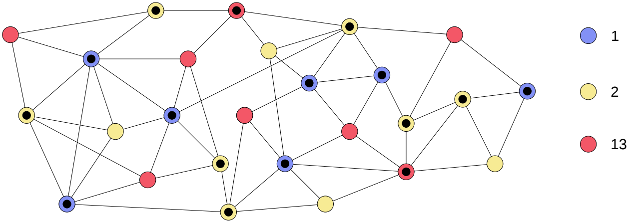
$\tilde{A}$ is **totally unimodular**

the bad: Min-Vertex-Cover



$$
\begin{array}{llcll}
\text{minimize} & \sum_{v \in V} \omega_v x_v & \Rightarrow OPT & & \\
\text{subject to} & x_u + x_v & \geq & 1 & \forall e = (u, v) \in E \\
 & x_v & \geq & 0 & \forall v \in V \\
 & x_v & \in & \mathbb{Z} &
\end{array}
$$

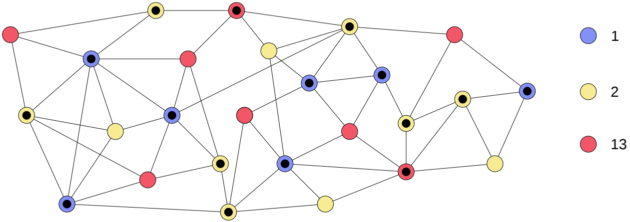
the bad: Min-Vertex-Cover



$$\begin{array}{lrcll} \text{minimize} & \sum_{v \in V} \omega_v x_v & \Rightarrow \mathbf{x}^\star & \text{relaxation} & \\ \text{subject to} & x_u + x_v & \geq & 1 & \forall e = (u, v) \in E \\ & x_v & \geq & 0 & \forall v \in V \end{array}$$

$$\hat{\mathbf{x}} := \text{ rounded } \mathbf{x}^\star$$

the bad: Min-Vertex-Cover



$$\begin{array}{lrcll} \text{minimize} & \sum_{v \in V} \omega_v x_v & & \Rightarrow \mathbf{x}^\star & \text{relaxation} \\ \text{subject to} & x_u + x_v & \geq & 1 & \forall e = (u,v) \in E \\ & x_v & \geq & 0 & \forall v \in V \end{array}$$

$$\hat{\mathbf{x}} := \text{ rounded } \mathbf{x}^\star$$

$$\hat{\mathbf{x}} \text{ is feasible}$$

$$\boldsymbol{\omega}^\mathsf{T}\hat{\mathbf{x}} = \sum_{v \in V} \omega_v \hat{x}_v = \sum_{\substack{v \in V \\ x_v \geq \frac{1}{2}}} \omega_v \hat{x}_v \leq 2 \sum_{\substack{v \in V \\ x_v \geq \frac{1}{2}}} \omega_v x_v \leq 2 \sum_{v \in V} \omega_v x_v = 2\boldsymbol{\omega}^\mathsf{T}\mathbf{x}^\star$$

the bad: Min-Vertex-Cover: half-integrality

basic solution is not a convex combination: $\mathbf{x} = t\mathbf{y} + (1-t)\mathbf{z}$

any non-half-integral $\mathbf{x}$

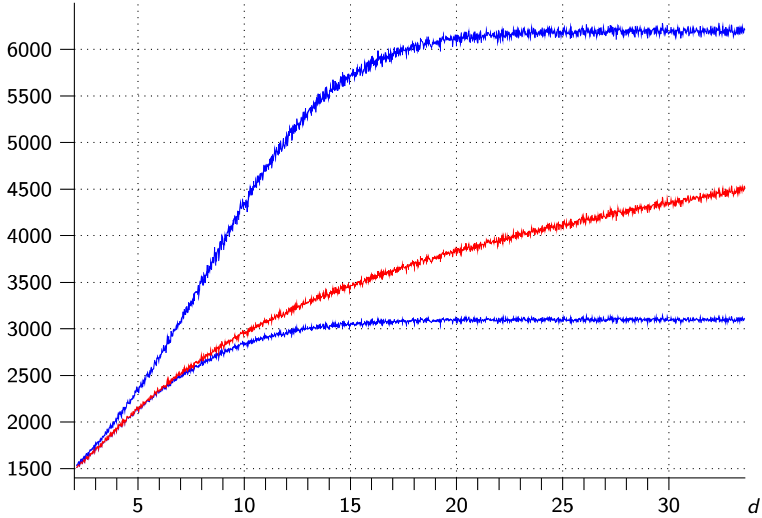
$$V_+ = \left\{ v \mid 0 < x_v < \frac{1}{2} \right\} \qquad V_- = \left\{ v \mid \frac{1}{2} < x_v < 1 \right\}$$

$$y_v = \begin{cases} x_v + \varepsilon, & \text{if } x_v \in V_+ \\ x_v - \varepsilon, & \text{if } x_v \in V_- \\ x_v, & \text{else} \end{cases} \qquad z_v = \begin{cases} x_v - \varepsilon, & \text{if } x_v \in V_+ \\ x_v + \varepsilon, & \text{if } x_v \in V_- \\ x_v, & \text{else} \end{cases}$$

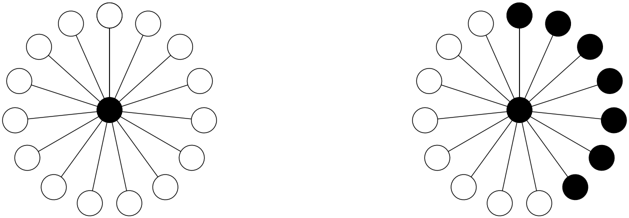
$$\mathbf{x} = \frac{1}{2}(\mathbf{y} + \mathbf{z})$$

the bad: Min-Vertex-Cover: experimental performance

random d-regular graphs, $n = 60$, $\omega = 1 + 2^i$ for random $i \in \{1, \ldots, 10\}$

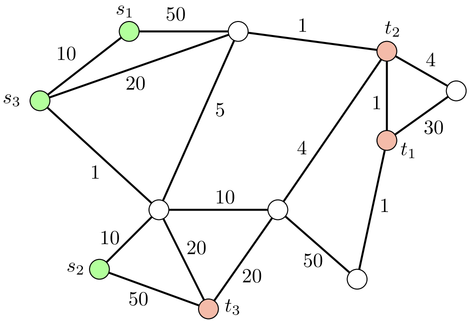


the ugly: Max-Independent-Set

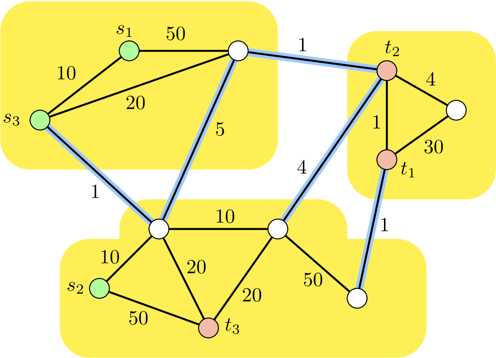


$$
\begin{array}{llcll}
\text{maximize} & \sum_{v \in V} \omega_v x_v & & & \\
\text{subject to} & x_u + x_v & \leq & 1 & \forall e = (u, v) \in E \\
 & x_v & \geq & 0 & \forall v \in V \\
 & x_v & \in & \mathbb{Z} &
\end{array}
$$

deterministic rounding: Min-Multi-Cut



deterministic rounding: Min-Multi-Cut



$$\begin{array}{llcll}
\text{minimize} & \sum_{e \in E} x_e \omega_e & & & \\
\text{subject to} & \sum_{e: e \in \pi} x_e & \geq & 1 & \forall i \in \{1, \ldots, k\},\ \forall \pi \in \mathcal{P}_{s_i, t_i} \\
& x_e & \geq & 0 & \forall e \in E \\
& x_e & \in & \mathbb{Z} &
\end{array}$$

Min-Multi-Cut: relaxation

relaxation

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} x_e \omega_e \\ \text{subject to} \quad & \sum_{e: e \in \pi} x_e \geq 1 \qquad \forall i \in \{1, \ldots, k\}, \ \forall \pi \in \mathcal{P}_{s_i, t_i} \\ & x_e \geq 0 \qquad \forall e \in E \end{aligned}$$

$x_e^\star$ length : $e = (u, v) \Rightarrow d(u, v) = x_e^\star$

ω_e cross section (area)

equivalent polynomial-sized formulation

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} x_e \omega_e \\ \text{subject to} \quad & x_e \geq p_v^{(i)} - p_u^{(i)} \qquad \forall i \in \{1, \ldots, k\}, \ \forall e \in E, \ e = (u, v) \\ & x_e \geq p_u^{(i)} - p_v^{(i)} \\ & p_{t_i}^{(i)} - p_{s_i}^{(i)} \geq 1 \qquad \forall i \in \{1, \ldots, k\} \end{aligned}$$

Min-Multi-Cut: $4\ln(2k)$-approximation

$$m \le 4\ln(2k) m_Q^\star \le 4\ln(2k) m^\star$$

ball with radius ρ $\quad \mathcal{B}_\rho(v) = \{u \in V_r' \mid d(u,v) \le \rho\}$

inner edges $\quad \mathcal{E}_\rho(v) = \{(w,z) \in E_r' \mid w, z \in \mathcal{B}_\rho(v)\}$

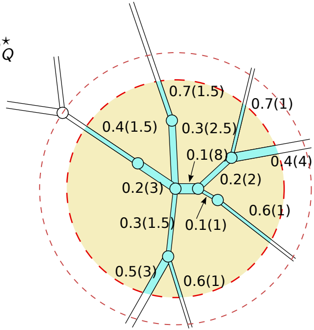
edge boundary $\quad \overline{\mathcal{E}}_\rho(v) = \{(w,z) \in E_r' - \mathcal{E}_\rho(v) \mid w \in \mathcal{B}_\rho(v) \vee z \in \mathcal{B}_\rho(v)\}$

ball volume $\quad V_\rho(v) = \frac{\Psi}{k} + \sum_{(w,z)\in\mathcal{E}_\rho(v)} \omega_{(w,z)} d(w,z) + \sum_{(w,z)\in\overline{\mathcal{E}}_\rho(v)} \omega_{(w,z)} (\rho - \min(d(v,w), d(v,z)))$

overall volume $\quad \Psi := \sum_{e=(u,v)\in E} \omega_e d(u,v) = m_Q^\star$

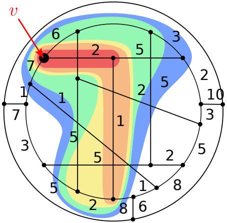
cost $\quad C_\rho(v) = \sum_{(w,z)\in\overline{\mathcal{E}}_\rho(v)} \omega_{(w,z)}$

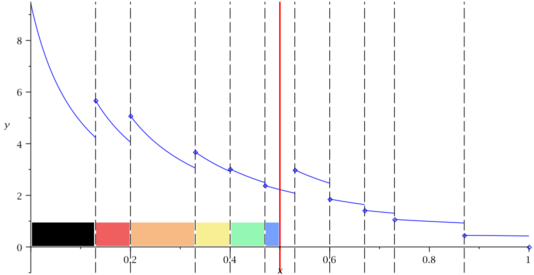
unit cost $\quad F_\rho(v) = \frac{C_\rho(v)}{V_\rho(v)}$



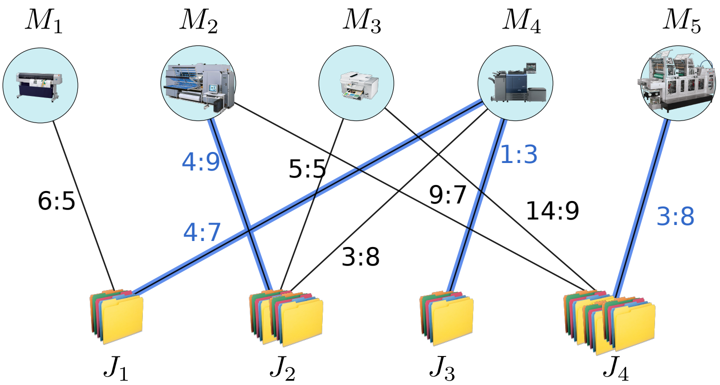
Min-Multi-Cut: $4\ln(2k)$-approximation

$$\forall v \exists \rho < 1/2 : \; F_\rho(v) \leq 2\ln(2k)$$



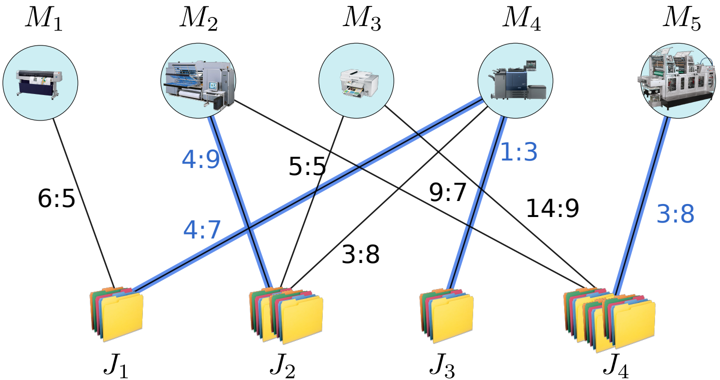


iterated rounding: Generalized-Assignment



$$
\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in E} x_{ij} c_{ij} & & \\
\text{subject to} \quad & \sum_{i\in M} x_{ij} = 1 & & \forall j \in J \\
& \sum_{j\in J} t_{ij} x_{ij} \leq T & & \forall i \in M \\
& x_{ij} \in \{0,1\} & & \forall i \in M,\ \forall j \in J
\end{aligned}
$$

iterated rounding: Generalized-Assignment



$$
\begin{array}{lrcll}
\text{minimize} & \sum_{(i,j)\in E} x_{ij}c_{ij} & & & \\
\text{subject to} & \sum_{i\in M} x_{ij} & = & 1 & \forall j \in J \\
 & \sum_{j\in J} t_{ij}x_{ij} & \leq & T_i & \forall i \in M' \\
 & x_{ij} & \geq & 0 & \forall i \in M,\ \forall j \in J
\end{array}
$$

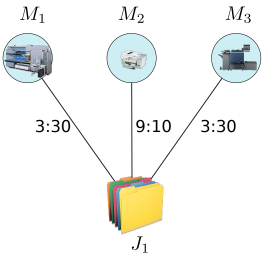
iterated rounding: Generalized-Assignment

optimal basic solution $\mathbf{x}$, s.t. $0 < x_{ij} < 1 \Rightarrow$ there is $i \in M'$, s.t. either

- $deg(i) \leq 1$ or
- $deg(i) = 2$ and $\sum_{j \in J} x_{ij} \geq 1$

basic solution $\mathbf{x}$, s.t. $x_{ij} > 0 \Rightarrow$ there is $M'' \subseteq M'$, $|M''| = |E| - |J|$:

$$\forall i \in M'' : \sum_{j \in J} t_{ij} x_{ij} = T_i$$



$$\max_{\mathbf{x} \in \mathbb{R}^6} \{\mathbf{c}^{\mathsf{T}} \mathbf{x} \mid A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0\}$$

$$\mathbf{c} = \begin{pmatrix} -3 \\ -9 \\ -3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad A = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 30 & 0 & 0 & 1 & 0 & 0 \\ 0 & 10 & 0 & 0 & 1 & 0 \\ 0 & 0 & 30 & 0 & 0 & 1 \end{array}\right) \qquad \mathbf{b} = \begin{pmatrix} 1 \\ T \\ T \\ T \end{pmatrix}$$

iterated rounding: Generalized-Assignment

optimal basic solution $\mathbf{x}$, s.t. $0 < x_{ij} < 1 \Rightarrow$ there is $i \in M'$, s.t. either

- $deg(i) \leq 1$ or
- $deg(i) = 2$ and $\sum_{j \in J} x_{ij} \geq 1$

basic solution $\mathbf{x}$, s.t. $x_{ij} > 0 \Rightarrow$ there is $M'' \subseteq M'$, $|M''| = |E| - |J|$:

$$\forall i \in M'' : \sum_{j \in J} t_{ij} x_{ij} = T_i$$

$$|J| + |M''| = |E| = \frac{\sum\limits_{j \in J} deg(j) + \sum\limits_{i \in M} deg(i)}{2} \geq \frac{\sum\limits_{j \in J} deg(j) + \sum\limits_{i \in M'} deg(i)}{2} \geq |J| + |M'| \geq |J| + |M''|$$

iterated rounding: Generalized-Assignment

1 $M' := M$, $\forall i:\ T_i := T$, $F := \emptyset$ (F is the set of assigned edges)

2 while $J \neq \emptyset$

- 3 let $\mathbf{x}$ be the optimal sol. of the relaxed program, perform one of the following:
- 4a if $\exists\ x_{ij} = 0$, remove edge (i, j) from G
- 4b if $\exists\ x_{ij} = 1$, $F := F \cup \{(i, j)\}$, $J := J \setminus \{j\}$, $T_i := T_i - t_{ij}$
- 4c else let $i \in M'$ s.t. $deg(i) \leq 1$ or $deg(i) = 2$ and $\sum_{j \in J} x_{ij} \geq 1$ $M' := M' \setminus \{i\}$

iterated rounding: Generalized-Assignment

1 $M' := M$, $\forall i:\ T_i := T$, $F := \emptyset$ (F is the set of assigned edges)
2 while $J \neq \emptyset$
3 let $\mathbf{x}$ be the optimal sol. of the relaxed program, perform one of the following:
4a if $\exists\ x_{ij} = 0$, remove edge (i, j) from G
4b if $\exists\ x_{ij} = 1$, $F := F \cup \{(i, j)\}$, $J := J \setminus \{j\}$, $T_i := T_i - t_{ij}$
4c else let $i \in M'$ s.t. $deg(i) \leq 1$ or $deg(i) = 2$ and $\sum_{j \in J} x_{ij} \geq 1$ $M' := M' \setminus \{i\}$

while $i \in M'$: $T_{i,\ell} + F_{i,\ell} \leq T$

machine j removed from M' in round ℓ (line 4c):

1. $deg(i) \leq 1$: time of $i \leq F_{i,\ell} + t_{ij} \leq T_{i,\ell} + F_{i,\ell} + t_{ij} \leq 2T$

2. $deg(i) = 2$ **and** $\sum_{j \in J} x_{ij} \geq 1$ time of $i \leq F_{i,\ell} + t_{ij_1} + t_{ij_2}$

$t_{ij_1}x_{ij_1} + t_{ij_2}x_{ij_2} \leq T_{i,\ell} \Rightarrow F_{i,\ell} \leq T - t_{ij_1}x_{ij_1} - t_{ij_2}x_{ij_2}$

time is at most

$$\begin{aligned} T - t_{ij_1}x_{ij_1} - t_{ij_2}x_{ij_2} + t_{ij_1} + t_{ij_2} &\leq \\ T + (1 - x_{ij_1})t_{ij_1} + (1 - x_{ij_2})t_{ij_2} &\leq \\ T + (1 - x_{ij_1})T + (1 - x_{ij_2})T &\leq \\ T(3 - x_{ij_1} - x_{ij_2}) &\leq 2T \end{aligned}$$

randomized rounding: Max-Sat

$$F = C_1 \wedge C_2 \wedge \cdots \wedge C_m \qquad C_i = l_{i,1} \vee l_{i,2} \vee \cdots \vee l_{i,k_{C_i}} \qquad \text{cost } \omega(C_i)$$

Maximize the cost of satisfied caluses.

Simple randomized algorithm `A1`: every variable with probability 1/2

- $Z := \sum_{i=1}^{m} \omega_i$, every clause has $s(C_i) \geq k$ literals
- $\mathrm{E}\left[\sum_{i=1}^{m} \omega_i X_i\right] = \sum_{i=1}^{m} \omega_i \mathrm{E}\left[X_i\right] = \sum_{i=1}^{m} \omega_i \Pr\left[X_i = 1\right] = \sum_{i=1}^{m} \omega_i \left(1 - 2^{-s(C_i)}\right) \geq (1 - 2^{-k})Z$
- derandomization

randomized rounding: Max-Sat

$$F = C_1 \wedge C_2 \wedge \cdots \wedge C_m \qquad C_i = l_{i,1} \vee l_{i,2} \vee \cdots \vee l_{i,k_{C_i}} \qquad \text{cost } \omega(C_i)$$

Maximize the cost of satisfied caluses.

Simple randomized algorithm `A1`: every variable with probability 1/2

- $Z := \sum_{i=1}^{m} \omega_i$, every clause has $s(C_i) \geq k$ literals
- $\mathrm{E}\left[\sum_{i=1}^{m} \omega_i X_i\right] = \sum_{i=1}^{m} \omega_i \mathrm{E}[X_i] = \sum_{i=1}^{m} \omega_i \Pr[X_i = 1] = \sum_{i=1}^{m} \omega_i \left(1 - 2^{-s(C_i)}\right) \geq (1 - 2^{-k})Z$
- derandomization

Algorithm `A2` for short caluses based on LP

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{m} \omega_i z_i \\ \text{subject to} \quad & \sum_{j \in C_i^+} p_j + \sum_{j \in C_i^-} (1 - p_j) \geq z_i \qquad \forall i = 1, \ldots, m \\ & z_i, p_i \in \mathbb{Z} \end{aligned}$$

randomized rounding: Max-Sat

$$F = C_1 \wedge C_2 \wedge \cdots \wedge C_m \qquad C_i = l_{i,1} \vee l_{i,2} \vee \cdots \vee l_{i,k_{C_i}} \qquad \text{cost } \omega(C_i)$$

Maximize the cost of satisfied caluses.

Simple randomized algorithm A1: every variable with probability 1/2

- $Z := \sum_{i=1}^{m} \omega_i$, every clause has $s(C_i) \geq k$ literals
- $\mathrm{E}\left[\sum_{i=1}^{m} \omega_i X_i\right] = \sum_{i=1}^{m} \omega_i \mathrm{E}\left[X_i\right] = \sum_{i=1}^{m} \omega_i \Pr\left[X_i = 1\right] = \sum_{i=1}^{m} \omega_i \left(1 - 2^{-s(C_i)}\right) \geq (1 - 2^{-k})Z$
- derandomization

Algorithm A2 for short caluses based on LP

$$\begin{array}{lrcll} \text{maximize} & \sum_{i=1}^{m} \omega_i z_i & & & \Rightarrow \quad z_i^\star, p_i^\star \ - \text{probability} \\ \text{subject to} & \sum_{j \in C_i^+} p_j + \sum_{j \in C_i^-} (1 - p_j) & \geq & z_i & \forall i = 1, \ldots, m \\ & z_i, p_i & \geq & 0 & \\ & z_i, p_i & \leq & 1 & \end{array}$$

randomized rounding: Max-Sat

Algorithm A2 for short caluses based on LP

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i=1}^{m} \omega_i z_i & \Rightarrow \quad & z_i^\star, p_i^\star \ - \text{probability} \\
\text{subject to} \quad & \sum_{j\in C_i^+} p_j + \sum_{j\in C_i^-} (1-p_j) & \geq \quad & z_i \qquad \forall i = 1,\dots,m \\
& z_i, p_i & \geq \quad & 0 \\
& z_i, p_i & \leq \quad & 1
\end{aligned}
$$

$$
\begin{aligned}
\Pr[X_i = 1] &= 1 - \prod_{j\in C_i^+} (1-p_j^\star) \cdot \prod_{j\in C_i^-} p_j^\star \\
&\geq 1 - \left(\frac{\sum_{j\in C_i^+} (1-p_j^\star) + \sum_{j\in C_i^-} p_j^\star}{s(C_i)} \right)^{s(C_i)} \\
&= 1 - \left(1 - \frac{\sum_{j\in C_i^+} p_j^\star + \sum_{j\in C_i^-} (1-p_j^\star)}{s(C_i)} \right)^{s(C_i)} \\
&\geq 1 - \left(1 - \frac{z_i^\star}{s(C_i)} \right)^{s(C_i)}
\end{aligned}
$$

randomized rounding: Max-Sat

$$\Pr[X_i = 1] \geq 1 - \left(1 - \frac{z_i^\star}{s(C_i)}\right)^{s(C_i)}$$

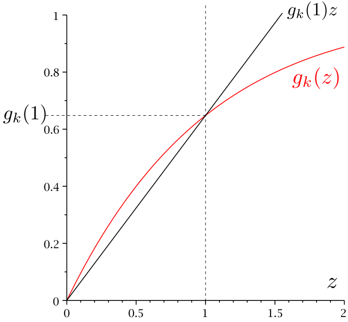
$$g_k(z) := 1 - \left(1 - \frac{z}{k}\right)^k$$

$$g_k'(z) = \left(1 - \frac{z}{k}\right)^{k-1}$$

$$g_k''(z) = -\frac{k-1}{k-z}\left(1 - \frac{z}{k}\right)^{k-2}$$

$$\beta(k) := g_k(1) = 1 - \left(1 - \frac{1}{k}\right)^k$$

$$\Pr[X_i = 1] \geq g_{s(C_i)}(1) \cdot z_i^\star = \beta(s(C_i)) \cdot z_i^\star$$

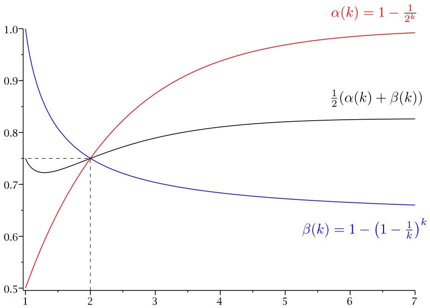


randomized rounding: Max-Sat

take better of the two algorithms

$$\texttt{A1det}(F) \geq \sum_{i=1}^{m} \omega_i \left(1 - \frac{1}{2^{s(C_i)}}\right) \qquad \texttt{A2det}(F) \geq \sum_{i=1}^{m} \omega_i \beta\left(s(C_i)\right) z_i^\star$$

$$\begin{aligned} & \max\{\texttt{A1det}(F), \texttt{A2det}(F)\} \\ \geq & \frac{1}{2}(\texttt{A1det}(F) + \texttt{A2det}(F)) \\ \geq & \frac{1}{2} \sum_{i=1}^{m} \omega_i \left(\left(1 - \frac{1}{2^{s(C_i)}}\right) + \beta(s(C_i)) z_i^\star \right) \\ \geq & \sum_{i=1}^{m} \omega_i z_i^\star \left(\frac{\left(1 - \frac{1}{2^{s(C_i)}}\right) + \beta(s(C_i))}{2} \right) \end{aligned}$$



LP duality

minimize $f(x_1, x_2, x_3) := 10x_1 + 3x_2 + 5x_3$

subject to

$$\begin{aligned} 6x_1 + x_2 - x_3 &\geq 2 \\ 2x_1 + 2x_2 + 6x_3 &\geq 8 \\ 6x_1 + 3x_2 + 5x_3 &= 30 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

LP duality

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maximize $g(y_1, y_2, y_3) := 2y_1 + 8y_2 + 30y_3$

subject to

$$\begin{aligned} 6y_1 + 2y_2 + 6y_3 &\leq 10 \\ y_1 + 2y_2 + 3y_3 &\leq 3 \\ -y_1 + 6y_2 + 5y_3 &\leq 5 \\ y_1, y_2 &\geq 0 \end{aligned}$$

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$$(P) : \min_{\mathbf{x} \in \mathbb{R}^n} \left\{ \mathbf{c}^\mathsf{T}\mathbf{x} \mid A\mathbf{x} \geq \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \right\} \qquad (D) : \max_{\mathbf{y} \in \mathbb{R}^m} \left\{ \mathbf{b}^\mathsf{T}\mathbf{y} \mid A^\mathsf{T}\mathbf{y} \leq \mathbf{c},\ \mathbf{y} \geq \mathbf{0} \right\}$$

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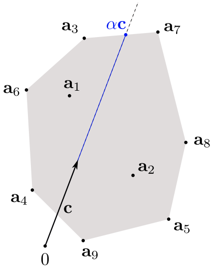
$$\mathbf{c}^\mathsf{T}\mathbf{x} \geq \mathbf{b}^\mathsf{T}\mathbf{y}$$

$$\mathbf{c}^\mathsf{T}\mathbf{x} \geq \left(A^\mathsf{T}\mathbf{y}\right)^\mathsf{T}\mathbf{x} = \mathbf{y}^\mathsf{T}A\mathbf{x} \geq \mathbf{y}^\mathsf{T}\mathbf{b}$$

LP duality

primal program		dual program	
minimize	$\mathbf{c}^T\mathbf{x}$	maximize	$\mathbf{b}^T\mathbf{y}$
i-th constraint	$\sum_{j=1}^{n} a_{ij}x_j = b_i$	i-th variable	$y_i \in \mathbb{R}$
i-th constraint	$\sum_{j=1}^{n} a_{ij}x_j \geq b_i$	i-th variable	$y_i \geq 0$
j-th variable	$x_j \in \mathbb{R}$	j-th constraint	$\sum_{i=1}^{m} a_{ij}y_i = c_j$
j-th variable	$x_j \geq 0$	j-th constraint	$\sum_{i=1}^{m} a_{ij}y_i \leq c_j$

strong duality – sketch



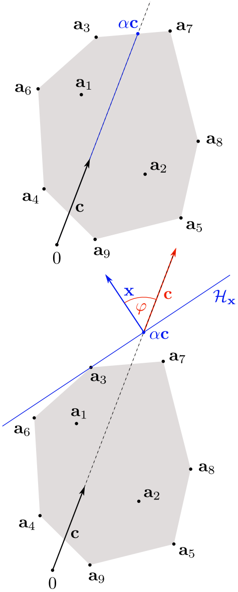
given $\mathbf{a_i}$, $\mathbf{c}$, find max. α, s.t. $\alpha\mathbf{c}$ is in convex hull

$$\alpha\mathbf{c} = z_1\mathbf{a}_1 + \cdots + z_n\mathbf{a}_n, \text{ s.t. } \sum_{i=1}^{n} z_i = 1$$

$$y_j = \frac{z_j}{\alpha} \qquad y_j \geq 0 \qquad \sum_{j=1}^{n} y_j = \frac{1}{\alpha} \qquad \alpha\mathbf{c} = \alpha \sum_{j=1}^{n} \mathbf{a}_j y_j.$$

$$\text{(P)} \quad \min_{\mathbf{y}\in\mathbb{R}^n} \left\{ \mathbf{1}^\mathsf{T}\mathbf{y} \mid A^\mathsf{T}\mathbf{y} = \mathbf{c},\ \mathbf{y} \geq \mathbf{0} \right\}$$

strong duality – sketch



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$$\text{(P)} \quad \min_{\mathbf{y}\in\mathbb{R}^n} \left\{ \mathbf{1}^\mathsf{T}\mathbf{y} \mid A^\mathsf{T}\mathbf{y} = \mathbf{c},\ \mathbf{y} \geq \mathbf{0} \right\}$$

$$\text{(D)} \quad \max_{\mathbf{x}\in\mathbb{R}^m} \left\{ \mathbf{c}^\mathsf{T}\mathbf{x} \mid A\mathbf{x} \leq \mathbf{1} \right\}$$

$$\mathcal{H}_\mathbf{x} := \{\mathbf{y} \mid \langle \mathbf{x}, \mathbf{y} \rangle = 1\} \qquad \alpha := \frac{1}{\mathbf{c}^\mathsf{T}\mathbf{x}}$$

strong duality – sketch

$$\min_{\mathbf{y}\in\mathbb{R}^n}\left\{\mathbf{1}^\mathsf{T}\mathbf{y} \mid A^\mathsf{T}\mathbf{y}=\mathbf{c},\ \mathbf{y}\geq\mathbf{0}\right\} = \max_{\mathbf{x}\in\mathbb{R}^m}\left\{\mathbf{c}^\mathsf{T}\mathbf{x} \mid A\mathbf{x}\leq\mathbf{1}\right\}$$

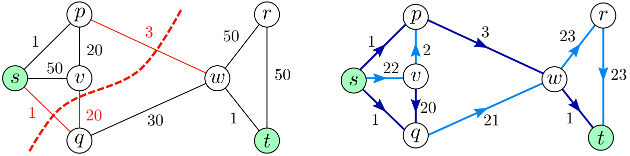
$$\text{(P)}\quad \min_{\mathbf{y}\in\mathbb{R}^n}\left\{\mathbf{b}^\mathsf{T}\mathbf{y} \mid A^\mathsf{T}\mathbf{y}=\mathbf{c},\ \mathbf{y}\geq\mathbf{0}\right\} \qquad \text{(D)}\quad \max_{\mathbf{x}\in\mathbb{R}^m}\left\{\mathbf{c}^\mathsf{T}\mathbf{x} \mid A\mathbf{x}\leq\mathbf{b}\right\}$$

$$\mathbf{a}_i' = \frac{1}{b_i}\mathbf{a}_i \qquad y_j' = b_j y_j$$

$$\mathbf{b}^\mathsf{T}\mathbf{y} = \sum_{j=1}^{n} b_j y_j = \mathbf{1}^\mathsf{T}\mathbf{y}' \qquad \sum_{j=1}^{m} a_{ji} y_j = \sum_{j=1}^{m} a_{ji}' b_i \frac{y_j'}{b_j}$$

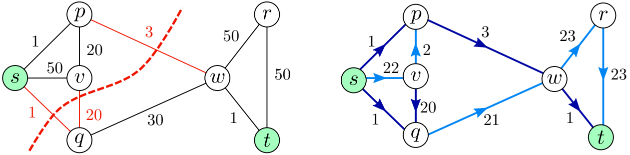
$$\text{(P)}\quad \min_{\mathbf{y}'\in\mathbb{R}^n}\left\{\mathbf{1}^\mathsf{T}\mathbf{y}' \mid A'\mathbf{y}'=\mathbf{c},\ \mathbf{y}'\geq\mathbf{0}\right\} \qquad \text{(D)}\quad \max_{\mathbf{x}'\in\mathbb{R}^m}\left\{\mathbf{c}^\mathsf{T}\mathbf{x}' \mid A'^\mathsf{T}\mathbf{x}'\leq\mathbf{1}\right\}$$

Max-Flow Min-Cut as dual problems



$$
\begin{aligned}
\text{maximize} \quad & f \\
\text{subject to} \quad & \sum_{u:(s,u)\in E} x_{su} - \sum_{u:(s,u)\in E} x_{us} - f = 0 \\
& \sum_{u:(t,u)\in E} x_{ut} - \sum_{u:(s,u)\in E} x_{tu} + f = 0 \\
& \sum_{u:(u,v)\in E} x_{vu} - \sum_{u:(u,v)\in E} x_{uv} = 0 \qquad \forall v \in V - \{s,t\} \\
& x_{uv} \le c_{uv} \qquad \forall (u,v) \in E \\
& x_{vu} \le c_{uv} \qquad \forall (u,v) \in E \\
& x_{uv} \ge 0 \qquad \forall (u,v) \in E
\end{aligned}
$$

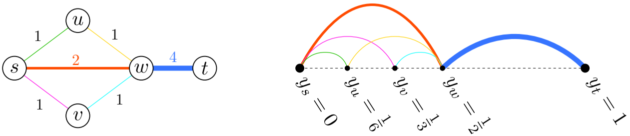
Max-Flow Min-Cut as dual problems



$$\text{minimize} \quad \sum_{(u,v)\in E} (z_{uv} + z_{vu})\, c_{uv}$$

$$\begin{array}{lrcll}
\text{subject to} & y_t - y_s & = & 1 & \\
& y_v - y_u + z_{uv} & \geq & 0 & \forall (u,v) \in E \\
& y_u - y_v + z_{vu} & \geq & 0 & \forall (u,v) \in E \\
& z_{uv}, z_{vu} & \geq & 0 & \forall (u,v) \in E
\end{array}$$

Max-Flow Min-Cut as dual problems



$$\text{minimize} \quad \sum_{(u,v)\in E} (z_{uv} + z_{vu})\, c_{uv}$$

$$\begin{array}{lrcll} \text{subject to} & y_t - y_s & = & 1 & \\ & y_v - y_u + z_{uv} & \geq & 0 & \forall (u,v) \in E \\ & y_u - y_v + z_{vu} & \geq & 0 & \forall (u,v) \in E \\ & z_{uv}, z_{vu} & \geq & 0 & \forall (u,v) \in E \end{array}$$

slackness conditions

$$(P): \min_{\mathbf{x}\in\mathbb{R}^n}\{\mathbf{c}^\mathsf{T}\mathbf{x} \mid A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq 0\} \qquad (D): \max_{\mathbf{y}\in\mathbb{R}^m}\{\mathbf{b}^\mathsf{T}\mathbf{y} \mid A^\mathsf{T}\mathbf{y} \leq \mathbf{c}, \mathbf{y} \geq 0\}$$

$$\mathbf{c}^\mathsf{T}\mathbf{x}^\star \geq \left(A^\mathsf{T}\mathbf{y}^\star\right)^\mathsf{T}\mathbf{x}^\star = \mathbf{y}^{\star\mathsf{T}}A\mathbf{x}^\star \geq \mathbf{y}^{\star\mathsf{T}}\mathbf{b}$$

slackness conditions

$$(P): \min_{\mathbf{x}\in\mathbb{R}^n} \{\mathbf{c}^\mathsf{T}\mathbf{x} \mid A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq 0\} \qquad (D): \max_{\mathbf{y}\in\mathbb{R}^m} \{\mathbf{b}^\mathsf{T}\mathbf{y} \mid A^\mathsf{T}\mathbf{y} \leq \mathbf{c}, \mathbf{y} \geq 0\}$$

$$\mathbf{c}^\mathsf{T}\mathbf{x}^\star \overset{(\clubsuit)}{=} \left(A^\mathsf{T}\mathbf{y}^\star\right)^\mathsf{T}\mathbf{x}^\star = \mathbf{y}^{\star\mathsf{T}}A\mathbf{x}^\star = \mathbf{y}^{\star\mathsf{T}}\mathbf{b}$$

$$(\clubsuit): \quad \sum_{j=1}^{n} c_j x_j^\star = \sum_{j=1}^{n} \left[A^\mathsf{T}\mathbf{y}^\star\right]_j x_j^\star$$

slackness conditions

$$(P): \min_{\mathbf{x}\in\mathbb{R}^n} \{\mathbf{c}^\mathsf{T}\mathbf{x} \mid A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq 0\} \qquad (D): \max_{\mathbf{y}\in\mathbb{R}^m} \{\mathbf{b}^\mathsf{T}\mathbf{y} \mid A^\mathsf{T}\mathbf{y} \leq \mathbf{c}, \mathbf{y} \geq 0\}$$

$$\mathbf{c}^\mathsf{T}\mathbf{x}^\star \overset{(\clubsuit)}{=} \left(A^\mathsf{T}\mathbf{y}^\star\right)^\mathsf{T}\mathbf{x}^\star = \mathbf{y}^{\star\mathsf{T}}A\mathbf{x}^\star = \mathbf{y}^{\star\mathsf{T}}\mathbf{b}$$

$$(\clubsuit): \quad \sum_{j=1}^{n} c_j x_j^\star = \sum_{j=1}^{n} \left[A^\mathsf{T}\mathbf{y}^\star\right]_j x_j^\star$$

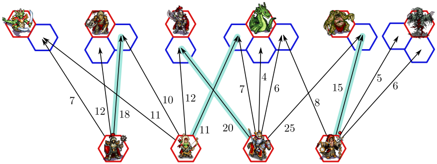
- primal slackness:

$$\forall\ 1 \leq j \leq n: \quad \text{either} \quad x_j = 0 \quad \text{or} \quad \sum_{i=1}^{m} a_{ij} y_i = c_j$$

- dual slackness:

$$\forall\ 1 \leq i \leq m: \quad \text{either} \quad y_i = 0 \quad \text{or} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i$$

primal-dual method

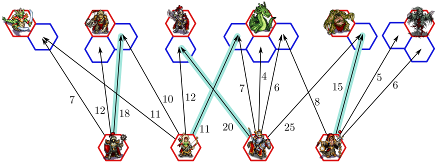


Max-W-Bipartite-Matching

$$\begin{array}{llll} \text{maximize} & \sum_{e\in E} \omega_e x_e & & \\ \text{subject to} & \sum_{\substack{e\in E\\ e=(v,w)}} x_e \le 1 & & \forall v \in V \\ & x_e \ge 0 & & \forall e \in E \\ & x_e \in \mathbb{Z} & & \end{array}$$

What happens for non-bipartite graphs?

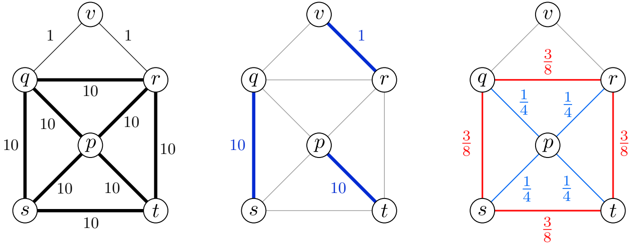
primal-dual method



Max-W-Bipartite-Matching

$$
\begin{array}{llccl}
\text{maximize} & \sum_{e\in E} \omega_e x_e & & & \\
\text{subject to} & \sum_{\substack{e\in E\\ e=(v,w)}} x_e & \leq & 1 & \forall v \in V \\
 & x_e & \geq & 0 & \forall e \in E \\
 & x_e & \in & \mathbb{Z} &
\end{array}
$$

What happens for non-bipartite graphs?



primal-dual method: Min-Perfect-Matching

$$\text{minimize} \sum_{e \in E} \omega_e x_e$$

$$\begin{aligned} \sum_{\substack{e \in E \\ e=(u,v)}} x_e &= 1 && \forall v \in V \\ x_e &\in \{0,1\} && \forall e \in E \end{aligned}$$

primal-dual method: Min-Perfect-Matching

$$\text{minimize} \sum_{e\in E} \omega_e x_e$$

$$\begin{aligned}
\sum_{e\in\delta(\{v\})} x_e &= 1 & \forall v \in V \\
\sum_{e\in\delta(S)} x_e &\geq 1 & \forall S \in \mathscr{S} \\
x_e &\geq 0 & \forall e \in E
\end{aligned}$$

$$\begin{aligned}
\delta(S) &:= \{e \in E \mid e = (u, v),\ u \in S,\ v \in V \setminus S\} \\
\mathscr{S} &:= \{S \subseteq V \mid\ |S| > 1,\ |S| \text{ odd } \}
\end{aligned}$$

primal-dual method: Min-Perfect-Matching

primal program

$$\text{minimize} \sum_{e \in E} \omega_e x_e$$

$$\begin{aligned} \sum_{e \in \delta(\{v\})} x_e &= 1 && \forall v \in V \\ \sum_{e \in \delta(S)} x_e &\geq 1 && \forall S \in \mathscr{S} \\ x_e &\geq 0 && \forall e \in E \end{aligned}$$

dual program

$$\text{maximize} \sum_{v \in V} r_v + \sum_{S \in \mathscr{S}} w_S$$

$$\begin{aligned} r_u + r_v + \sum_{\substack{S \in \mathscr{S} \\ e \in \delta(S)}} w_S &\leq \omega_e && \forall e = (u, v) \in E \\ w_S &\geq 0 && \forall S \in \mathscr{S} \end{aligned}$$

$$\begin{aligned} \delta(S) &:= \{e \in E \mid e = (u, v),\ u \in S,\ v \in V \setminus S\} \\ \mathscr{S} &:= \{S \subseteq V \mid \ |S| > 1,\ |S| \text{ odd } \} \end{aligned}$$

primal-dual method: Min-Perfect-Matching

$$\begin{aligned} \text{minimize} \quad & \sum_{e\in E} \omega_e x_e \\ \sum_{e\in\delta(\{v\})} x_e &= 1 \quad \forall v \in V \\ \sum_{e\in\delta(S)} x_e &\geq 1 \quad \forall S \in \mathscr{S} \\ x_e &\geq 0 \quad \forall e \in E \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & \sum_{v\in V} r_v + \sum_{S\in\mathscr{S}} w_S \\ r_u + r_v + \sum_{\substack{S\in\mathscr{S}\\ e\in\delta(S)}} w_S &\leq \omega_e \quad \forall e=(u,v)\in E \\ w_S &\geq 0 \quad \forall S \in \mathscr{S} \end{aligned}$$

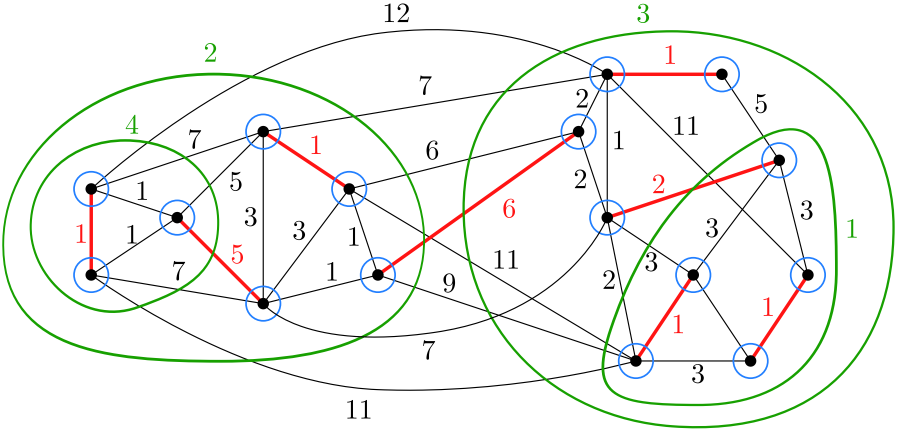
$$\sum_{e\in E} x_e\omega_e \overset{\clubsuit}{\geq} \sum_{\substack{e\in E\\ e=(u,v)}} x_e(r_u + r_v + \sum_{\substack{S\in\mathscr{S}\\ e\in\delta(S)}} w_S) = \sum_{v\in V}(r_v \sum_{e\in\delta(\{v\})} x_e) + \sum_{S\in\mathscr{S}} w_S \left(\sum_{e\in\delta(S)} x_e\right) \overset{\diamondsuit}{\geq} \sum_{v\in V} r_v + \sum_{S\in\mathscr{S}} w_S$$

slackness conditions

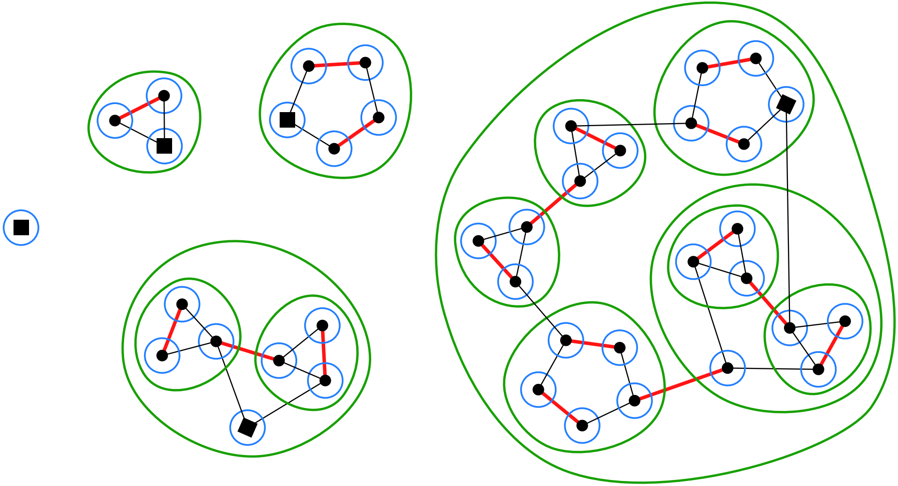
S1(♣) $\forall e=(u,v)\in E: \quad x_e > 0 \Rightarrow r_u + r_v + \sum_{\substack{S\in\mathscr{S}\\ e\in\delta(S)}} w_S = \omega(e)$

S2(◇) $\forall S \in \mathscr{S}: \quad w_S > 0 \Rightarrow \sum_{e\in\delta(S)} x_e = 1$

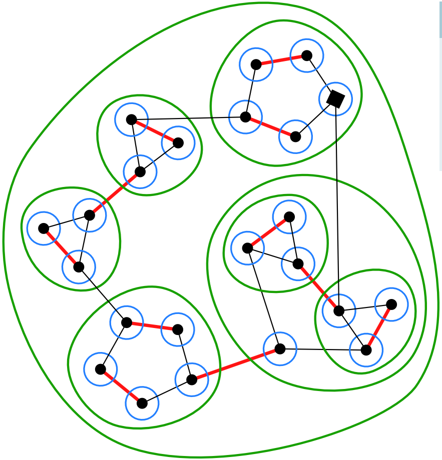
primal-dual method: Min-Perfect-Matching



primal-dual method: Min-Perfect-Matching



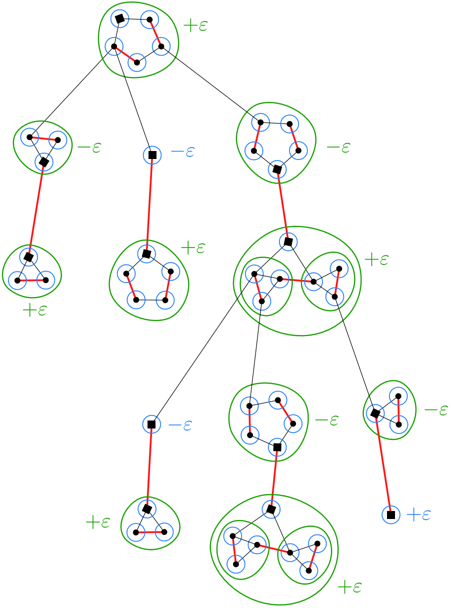
primal-dual method: Min-Perfect-Matching



flower

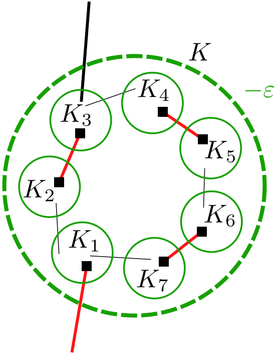
- non-zero outer bubble w_S
- a root vertex
- odd number of inner flowers paired in dumbells
- cyclically connected with full edges

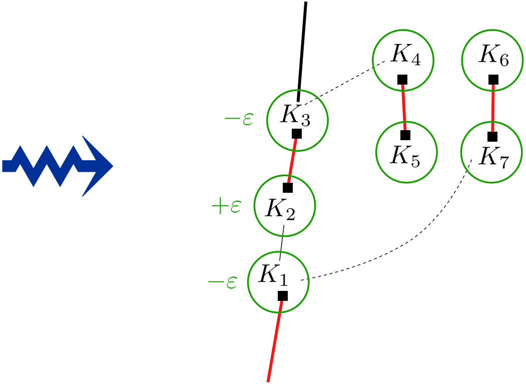
primal-dual method: Min-Perfect-Matching



primal-dual method: Min-Perfect-Matching

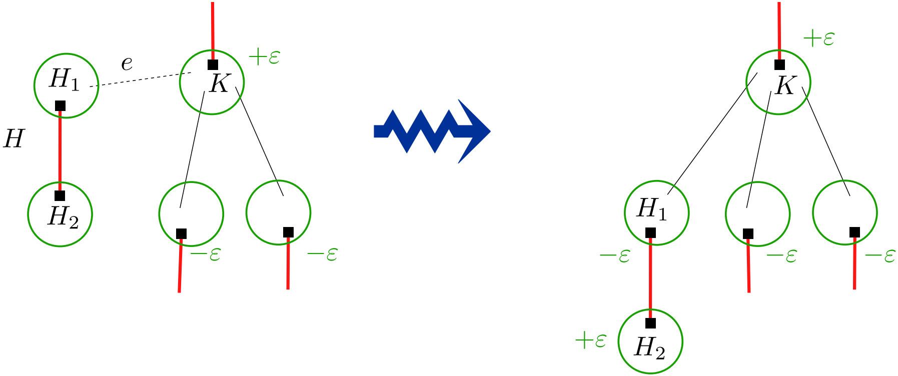
(P1) Green bubble drops to zero.





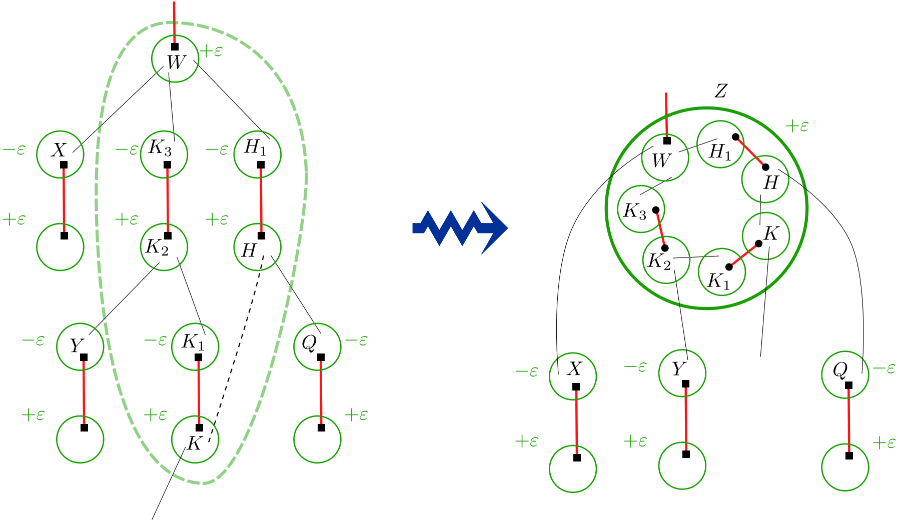
primal-dual method: Min-Perfect-Matching

(P2) Edge e between an even-level flower K and a dumbell H is filled.



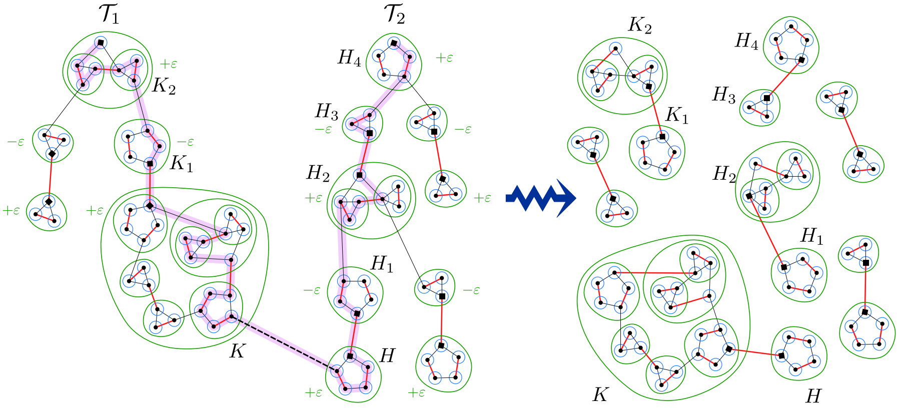
primal-dual method: Min-Perfect-Matching

(P3) Edge between two flowers K and H in the same tree is filled.



primal-dual method: Min-Perfect-Matching

(P4) Edge e between two flowers K and H in different trees $\mathcal{T}_1$ and $\mathcal{T}_2$ is filled.



∀ v ∃ alternating path to the flower-root, starting by matching egde, and ending by non-matching

primal-dual method: Min-Perfect-Matching

(P1) Green bubble drops to zero.

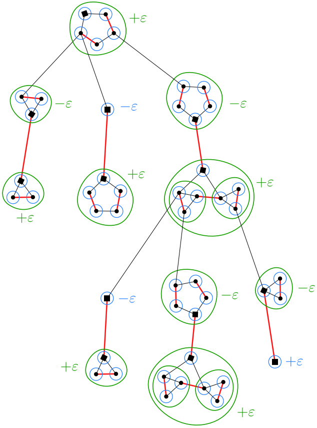
(P2) Edge e between an even-level flower K and a dumbell H is filled.

(P3) Edge between two flowers K and H in the same tree is filled.

(P4) Edge e between two flowers K and H in different trees $\mathcal{T}_1$ and $\mathcal{T}_2$ is filled.

$O(n^2)$ iterations

- $O(n)$ iterations of (P4)
- between two interations of (P4):
  - $O(n)$ bubbles overall
  - safe bubble: sometime was outer bubble on "+" level
  - safe bubble is never destroyed
  - (P1), (P2), (P3) create at least one safe bubble



relaxed slackness conditions

$$(P) : \min_{\mathbf{x} \in \mathbb{R}^n} \{\mathbf{c}^\mathsf{T}\mathbf{x} \mid A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq 0\} \qquad (D) : \max_{\mathbf{y} \in \mathbb{R}^m} \{\mathbf{b}^\mathsf{T}\mathbf{y} \mid A^\mathsf{T}\mathbf{y} \leq \mathbf{c}, \mathbf{y} \geq 0\}$$

- primal slackness:

$$\forall\ 1 \leq j \leq n : \quad \text{either} \quad x_j = 0 \quad \text{or} \quad \sum_{i=1}^{m} a_{ij} y_i = c_j$$

- dual slackness:

$$\forall\ 1 \leq i \leq m : \quad \text{either} \quad y_i = 0 \quad \text{or} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i$$

relaxed slackness conditions

$$(P): \min_{\mathbf{x}\in\mathbb{R}^n} \{\mathbf{c}^\mathsf{T}\mathbf{x} \mid A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq 0\} \qquad (D): \max_{\mathbf{y}\in\mathbb{R}^m} \{\mathbf{b}^\mathsf{T}\mathbf{y} \mid A^\mathsf{T}\mathbf{y} \leq \mathbf{c}, \mathbf{y} \geq 0\}$$

- primal slackness:

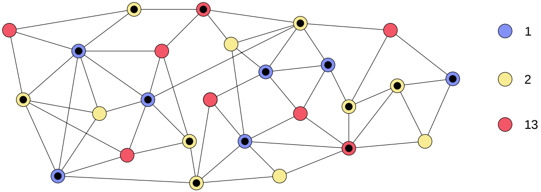
$$\forall\ 1 \leq j \leq n : \quad \text{either} \quad x_j = 0 \quad \text{or} \quad c_j/\alpha \leq \sum_{i=1}^{m} a_{ij} y_i \leq c_j$$

- dual slackness:

$$\forall\ 1 \leq i \leq m : \quad \text{either} \quad y_i = 0 \quad \text{or} \quad b_i \leq \sum_{j=1}^{n} a_{ij} x_j \leq \beta b_i$$

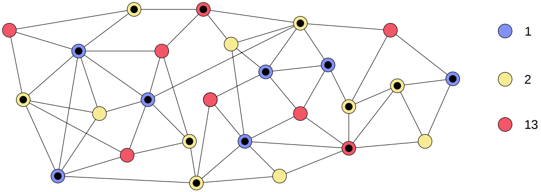
$$\mathbf{c}^\mathsf{T}\mathbf{x} = \sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} \alpha \left(\sum_{i=1}^{m} a_{ij} y_i\right) x_j \leq \alpha \sum_{i=1}^{m} y_i \left(\sum_{j=1}^{n} a_{ij} x_j\right) \leq \alpha\beta \sum_{i=1}^{m} y_i b_i = \alpha\beta\mathbf{b}^\mathsf{T}\mathbf{y}$$

Min-Vertex-Cover revisited



$$
\begin{array}{llcll}
\text{minimize} & \sum_{v \in V} \omega_v x_v & & & \\
\text{subject to} & x_u + x_v & \geq & 1 & \forall e = (u, v) \in E \\
 & x_v & \geq & 0 & \forall v \in V \\
 & x_v & \in & \mathbb{Z} &
\end{array}
$$

Min-Vertex-Cover revisited



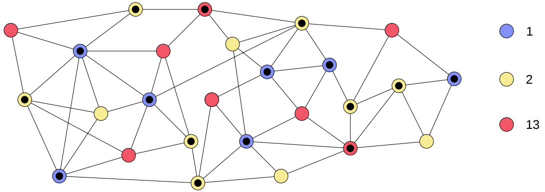
primal program

$$\begin{array}{lrcll} \text{minimize} & \sum_{v \in V} \omega_v x_v & & & \\ \text{subject to} & x_u + x_v & \geq & 1 & \forall e \in E \\ & x_v & \geq & 0 & \forall v \in V \end{array}$$

dual program

$$\begin{array}{lrcll} \text{maximize} & \sum_{e \in E} y_e & & & \\ \text{subject to} & \sum_{\substack{e \in E \\ e=(u,v)}} y_e & \leq & \omega_u & \forall u \in V \\ & y_e & \geq & 0 & \forall e \in E \end{array}$$

Min-Vertex-Cover revisited



primal program

$$\begin{array}{lrcll} \text{minimize} & \sum_{v\in V} \omega_v x_v & & & \\ \text{subject to} & x_u + x_v & \geq & 1 & \forall e \in E \\ & x_v & \geq & 0 & \forall v \in V \end{array}$$

dual program

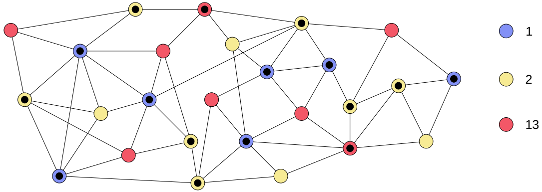
$$\begin{array}{lrcll} \text{maximize} & \sum_{e\in E} y_e & & & \\ \text{subject to} & \sum_{\substack{e\in E\\ e=(u,v)}} y_e & \leq & \omega_u & \forall u \in V \\ & y_e & \geq & 0 & \forall e \in E \end{array}$$

slackness conditions

S1 $\forall v \in V: \quad x_v > 0 \Rightarrow \sum_{\substack{e\in E\\ e=(u,v)}} y_e = \omega_u$

S2 $\forall e = (u,v) \in E: \quad y_e > 0 \Rightarrow x_u + x_v = 1$

Min-Vertex-Cover revisited



primal program

$$\begin{array}{llll} \text{minimize} & \sum_{v\in V} \omega_v x_v & & \\ \text{subject to} & x_u + x_v \geq 1 & \forall e \in E \\ & x_v \geq 0 & \forall v \in V \end{array}$$

dual program

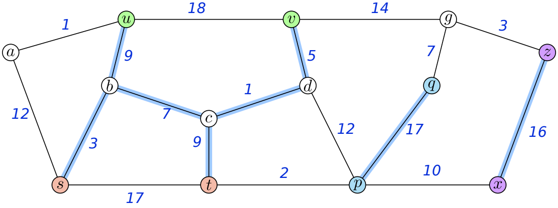
$$\begin{array}{llll} \text{maximize} & \sum_{e\in E} y_e & & \\ \text{subject to} & \sum_{\substack{e\in E\\ e=(u,v)}} y_e \leq \omega_u & \forall u \in V \\ & y_e \geq 0 & \forall e \in E \end{array}$$

slackness conditions

S1 $\forall v \in V : \quad x_v > 0 \Rightarrow \sum_{\substack{e\in E\\ e=(u,v)}} y_e = \omega_u$

S2 $\forall e = (u,v) \in E : \quad y_e > 0 \Rightarrow x_u + x_v \leq 2$

weaker than relaxed slackness: Min-Steiner-Forest



connectivity requirements $r(u, v) \in \{0, 1\}$

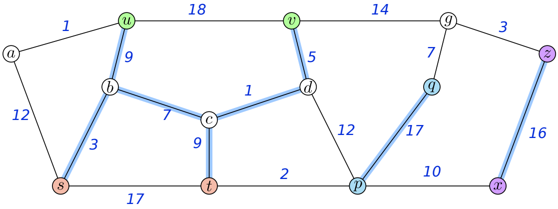
hungry set S: $f(S) = 1$

$$\exists u \in S, v \in V \setminus S : \quad r(u, v) = 1$$

Lemma

If there is an outgoing edge from each hungry set $\Rightarrow$ feasible solution

weaker than relaxed slackness: Min-Steiner-Forest



primal program

$$\text{minimize} \quad \sum_{e \in E} \omega_e x_e$$

$$\sum_{e \in \delta(S)} x_e \geq f(S) \quad \forall S \subseteq V$$

$$x_e \geq 0 \quad \forall e \in E$$

dual program

$$\text{maximize} \quad \sum_{S \subseteq V} y_s f(S)$$

$$\sum_{S : e \in \delta(S)} y_S \leq \omega_e \quad \forall e \in E$$

$$y_S \geq 0 \quad \forall S \subseteq V$$

weaker than relaxed slackness: Min-Steiner-Forest

primal program

$$\begin{aligned}
\text{minimize} \quad & \sum_{e\in E} \omega_e x_e \\
& \sum_{e\in\delta(S)} x_e \geq f(S) \quad \forall S\subseteq V \\
& x_e \geq 0 \quad \forall e\in E
\end{aligned}$$

dual program

$$\begin{aligned}
\text{maximize} \quad & \sum_{S\subseteq V} y_s f(S) \\
& \sum_{S:e\in\delta(S)} y_S \leq \omega_e \quad \forall e\in E \\
& y_S \geq 0 \quad \forall S\subseteq V
\end{aligned}$$

slackness conditions

S1 $\forall e\in E: \quad x_e > 0 \Rightarrow \sum_{S:e\in\delta(S)} y_S = \omega_e$

S2 $\forall S\subseteq V: \quad y_S > 0 \Rightarrow \sum_{e\in\delta(S)} x_e = f(S)$

weaker than relaxed slackness: Min-Steiner-Forest

primal program

$$\begin{array}{llll} \text{minimize} & \sum_{e\in E} \omega_e x_e & & \\ & \sum_{e\in\delta(S)} x_e \geq & f(S) & \forall S \subseteq V \\ & x_e \geq & 0 & \forall e \in E \end{array}$$

dual program

$$\begin{array}{llll} \text{maximize} & \sum_{S\subseteq V} y_s f(S) & & \\ & \sum_{S:e\in\delta(S)} y_S \leq & \omega_e & \forall e \in E \\ & y_S \geq & 0 & \forall S \subseteq V \end{array}$$

slackness conditions

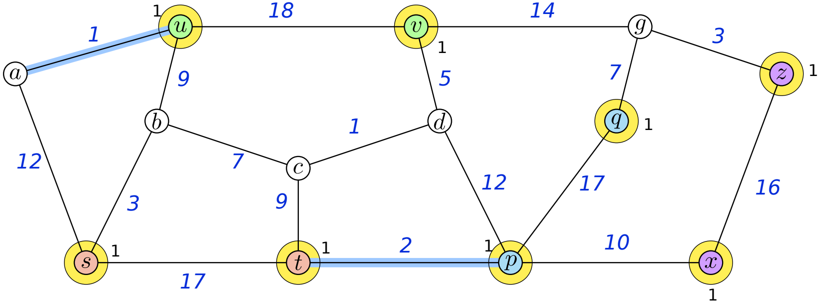
S1 $\forall e \in E: \quad x_e > 0 \Rightarrow \sum_{S:e\in\delta(S)} y_S = \omega_e$

S2 $\forall S \subseteq V: \quad y_S > 0 \Rightarrow \sum_{e\in\delta(S)} x_e \leq 2f(S)$

weaker than relaxed slackness: Min-Steiner-Forest

unhappy set: hungry, but no outgoing edge

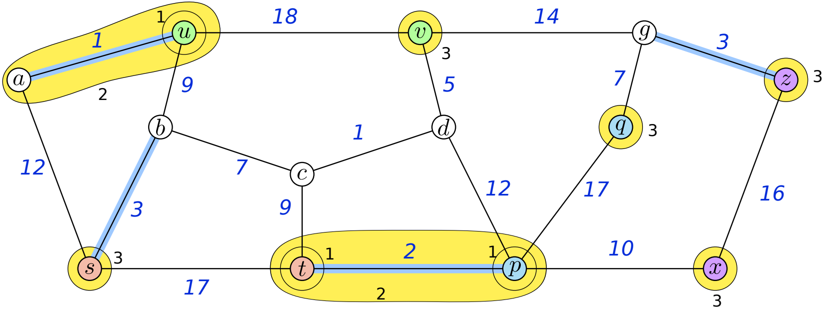
increase minimal (w.r.t. inclusion) unhappy sets



weaker than relaxed slackness: Min-Steiner-Forest

unhappy set: hungry, but no outgoing edge

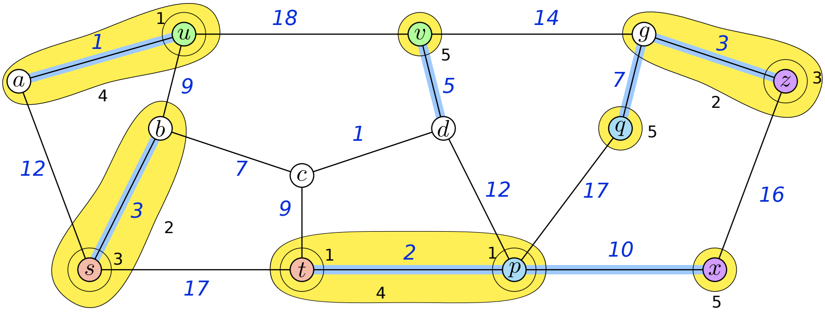
increase minimal (w.r.t. inclusion) unhappy sets (connected components)



weaker than relaxed slackness: Min-Steiner-Forest

unhappy set: hungry, but no outgoing edge

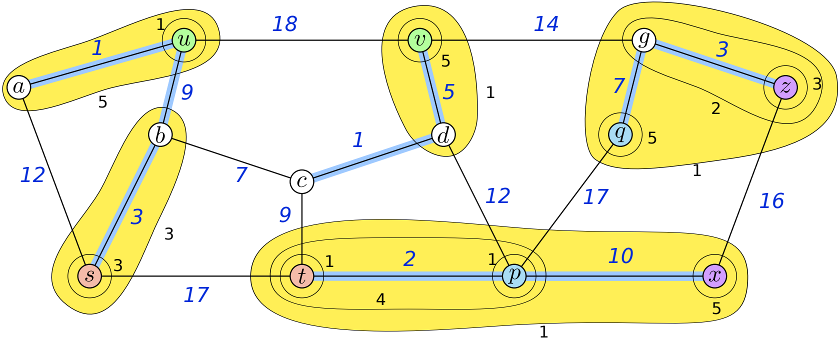
increase minimal (w.r.t. inclusion) unhappy sets (connected components)



weaker than relaxed slackness: Min-Steiner-Forest

unhappy set: hungry, but no outgoing edge

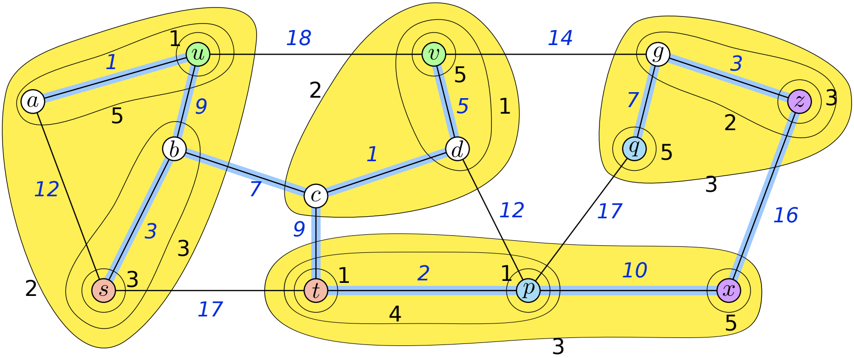
increase minimal (w.r.t. inclusion) unhappy sets (connected components)



weaker than relaxed slackness: Min-Steiner-Forest

unhappy set: hungry, but no outgoing edge

increase minimal (w.r.t. inclusion) unhappy sets (connected components)

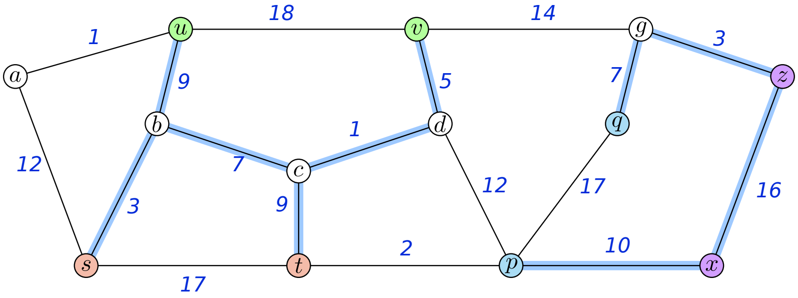


postprocessing: prune result

weaker than relaxed slackness: Min-Steiner-Forest

unhappy set: hungry, but no outgoing edge

increase minimal (w.r.t. inclusion) unhappy sets (connected components)



postprocessing: prune result

weaker than relaxed slackness: Min-Steiner-Forest

algorithm

$F := \emptyset$, $y_{\{v\}} := 0$ for all $v \in V$

while there is an unhappy connected component induced by edges from F

    increase y_S for all S corresponding to unhappy sets from F until some edge e is tight
    $F := F \cup e$

$F' := F$

for each edge $e \in F$

    if $F - \{e\}$ is feasible solution $F' := F' - \{e\}$

Observation: F is acyclic $\Rightarrow$ pruning maintains feasibility

weaker than relaxed slackness: Min-Steiner-Forest

algorithm

1 $F := \emptyset$, $y_{\{v\}} := 0$ for all $v \in V$

2 while there is an unhappy connected component induced by edges from F

  3 increase y_S for all S corresponding to unhappy sets from F until some edge e is tight

  4 $F := F \cup e$

5 $F' := F$

6 for each edge $e \in F$

  7 if $F - \{e\}$ is feasible solution $F' := F' - \{e\}$

primal program

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} \omega_e x_e \\ \sum_{e \in \delta(S)} x_e &\geq f(S) \quad \forall S \subseteq V \\ x_e &\geq 0 \quad \forall e \in E \end{aligned}$$

dual program

$$\begin{aligned} \text{maximize} \quad & \sum_{S \subseteq V} y_S f(S) \\ \sum_{S : e \in \delta(S)} y_S &\leq \omega_e \quad \forall e \in E \\ y_S &\geq 0 \quad \forall S \subseteq V \end{aligned}$$

Theorem

$$\sum_{e \in F'} \omega_e \leq 2 \sum_{S \subseteq V} y_S f(S)$$

weaker than relaxed slackness: Min-Steiner-Forest

Theorem

$$\sum_{e\in F'} \omega_e \leq 2 \sum_{S\subseteq V} y_S f(S) = 2 \sum_{S\subseteq V} y_S$$

$$\sum_{e\in F'} \omega_e = \sum_{e\in F'} \left(\sum_{S:e\in\delta(S)} y_S \right) = \sum_{S\subseteq V} \deg_{F'}(S) y_S \overset{?}{\leq} 2 \sum_{S\subseteq V} y_S$$

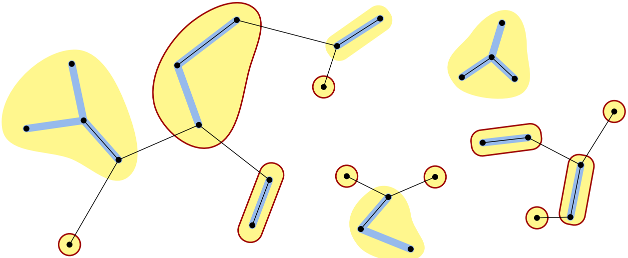
weaker than relaxed slackness: Min-Steiner-Forest

Theorem

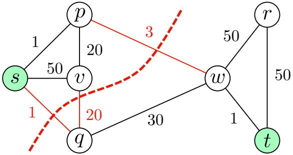
$$\sum_{e\in F'} \omega_e \le 2 \sum_{S\subseteq V} y_S f(S) = 2 \sum_{S\subseteq V} y_S$$

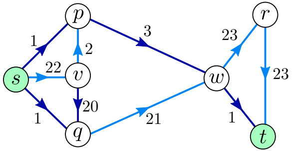
$$\sum_{e\in F'} \omega_e = \sum_{e\in F'} \left(\sum_{S: e\in\delta(S)} y_S \right) = \sum_{S\subseteq V} \deg_{F'}(S) y_S \overset{?}{\le} 2 \sum_{S\subseteq V} y_S$$

$$\Delta \left(\sum_{S\in\mathscr{S}_\ell} \deg_{F'}(S) \right) \le 2\Delta|\mathscr{S}_\ell|$$

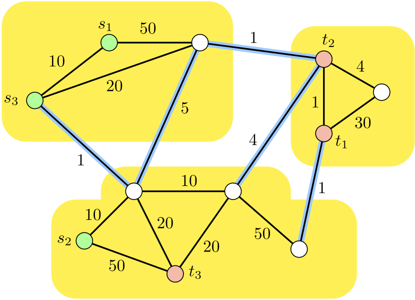


variations on Min-Cut: a theme

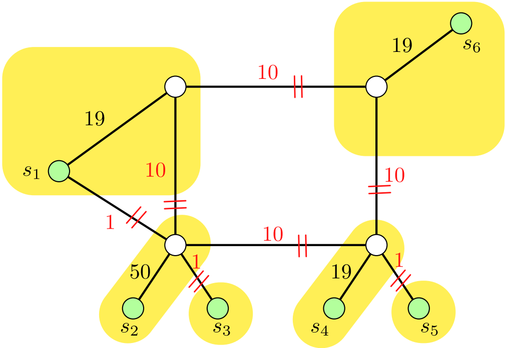




4 ln(2k)-approximation for Min-Multi-Cut



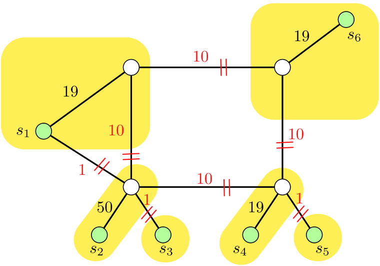
first variation: Min-Multiway-Cut

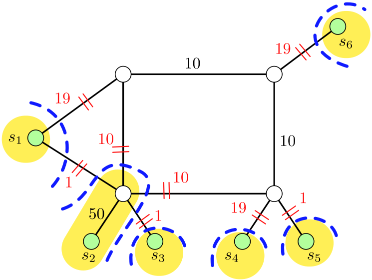


Given are $G = (V, E)$, $\omega : E \mapsto \mathbb{R}^+$, and k vertices $s_1, \ldots, s_k$.

Find a minimal cut that disconnects all pairs s_j, s_j, $i \neq j$.

first variation: Min-Multiway-Cut





$$ALG \leq \sum_{\substack{i=1 \\ i \neq j}}^{k} \partial(D_i) = \sum_{i=1}^{k} \partial(D_i) - \partial(D_j) \leq \left(1 - \frac{1}{k}\right) \sum_{i=1}^{k} \partial(D_i) \leq \left(1 - \frac{1}{k}\right) \sum_{i=1}^{k} \partial(C_i) = 2\left(1 - \frac{1}{k}\right) OPT$$

minimal s_i-separating cut

component of the optimal solution

second variation: Min-k-Cut

Given are $G = (V, E)$, $\omega : E \mapsto \mathbb{R}^+$, and a number k.

Find a minimal cut that results in $\geq k$ components.

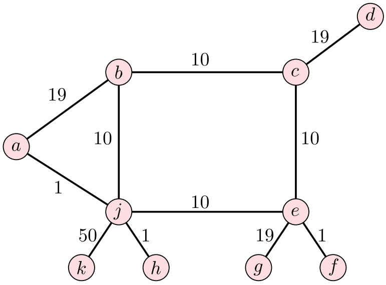
Gomory-Hu trees

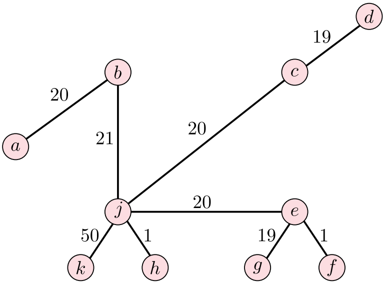
Given $G = (V, E)$, $\omega : E \mapsto \mathbb{R}^+$.

size of the cut in G defined by removing e' from T

GHT is a tree $T = (V, E')$ with $\omega' : E' \mapsto \mathbb{R}^+$:

1. $\forall e' \in E' : \ \omega'(e') = \partial_G(cut_T(e'))$
2. $\forall u, v \in V : \ f_G(u, v) = f_T(u, v)$





Gomory-Hu trees

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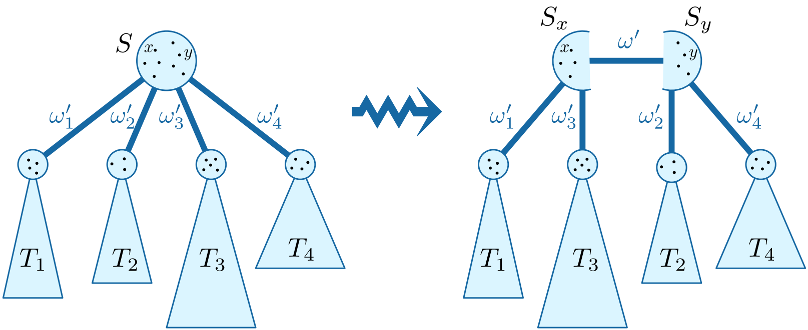
2. $\forall u, v \in V : \ f_G(u, v) = f_T(u, v)$

Algorithm: splitting bags



Gomory-Hu trees

Given $G = (V, E)$, $\omega : E \mapsto \mathbb{R}^+$.

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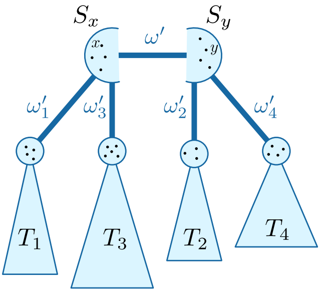
Algorithm: splitting bags



Invariant

$\forall \ e = \left(S_i^{(t)}, S_J^{(t)}\right)$

$M : \{v \in V \mid \exists S \in cut_{T^{(t)}}(e) : \ v \in S\}$

$\exists$ 2 witnesses $x \in S_i^{(t)}$, $y \in S_j^{(t)}$:

- $\omega'(e) = f_G(x, y)$
- M is minimal $x - y$ cut in G.

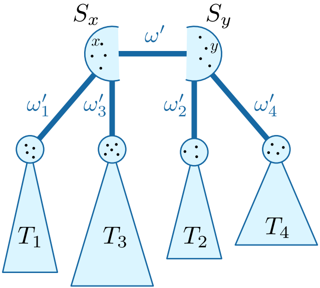
Gomory-Hu trees

Given $G = (V, E)$, $\omega : E \mapsto \mathbb{R}^+$.

size of the cut in G defined by removing e' from T

GHT is a tree $T = (V, E')$ with $\omega' : E' \mapsto \mathbb{R}^+$:

1. $\forall e' \in E' : \omega'(e') = \partial_G(cut_T(e'))$
2. $\forall u, v \in V : f_G(u, v) = f_T(u, v)$



Invariant

$\forall\ e = \left(S_i^{(t)}, S_J^{(t)}\right)$

$M : \{v \in V \mid \exists S \in cut_{T^{(t)}}(e) : \ v \in S\}$

$\exists$ 2 witnesses $x \in S_i^{(t)}$, $y \in S_j^{(t)}$:

- $\omega'(e) = f_G(x, y)$
- M is minimal $x - y$ cut in G.

after finishing $\Rightarrow$ GHT

1. $\forall e' \in E' : \omega'(e') = \partial_G(cut_T(e'))$ $\leftarrow$ from the invariant
2. $\forall u, v \in V : f_G(u, v) = f_T(u, v)$
   - either $(u, v) \in E'$ $\leftarrow$ from the invariant
   - or $(u, v) \notin E'$ $\leftarrow$???

Gomory-Hu trees

Given $G = (V, E)$, $\omega : E \mapsto \mathbb{R}^+$.

GHT is a tree $T = (V, E')$ with $\omega' : E' \mapsto \mathbb{R}^+$:

1. $\forall e' \in E' : \omega'(e') = \partial_G(cut_T(e'))$
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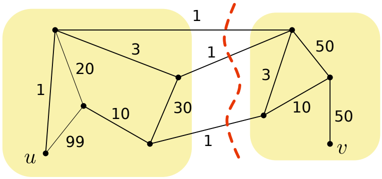

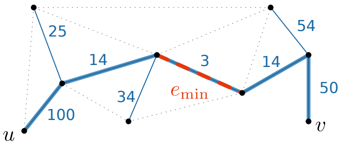

$$f_T(u, v) = \omega'(e_{\min}) = \partial_G(cut_T(e_{\min})) \geq f_G(u, v)$$

Gomory-Hu trees

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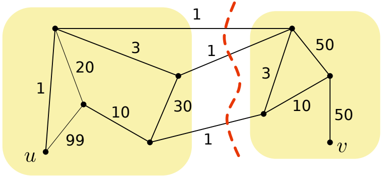
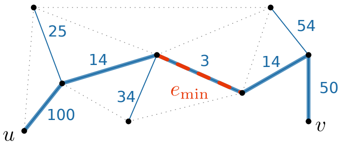


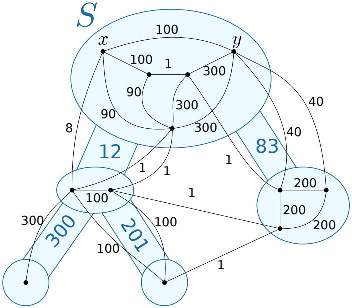


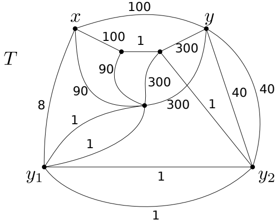
$$f_G(u, v) \geq \min\{\omega'(w_0, w_1), \ldots, \omega'(w_{z-1}, w_z)\} = \omega'(e_{\min}) = f_T(u, v)$$

Lemma

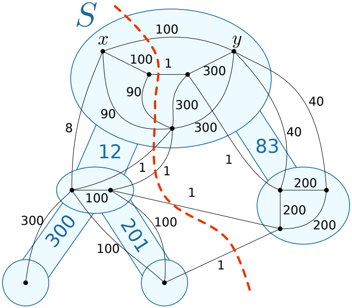
$$f_G(v_1, v_z) \geq \min\{f_G(v_1, v_2), f_G(v_2, v_3), \ldots, f_G(v_{z-1}, v_z)\}$$

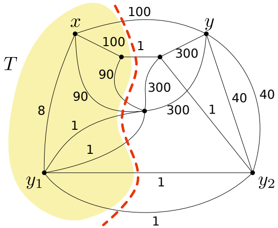
Gomory-Hu trees: how the splitting is made





Gomory-Hu trees: how the splitting is made





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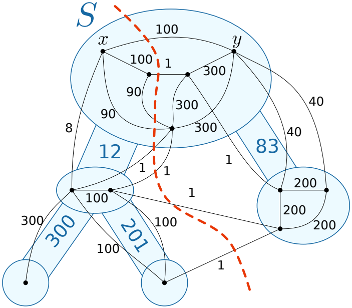
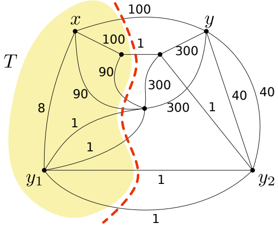
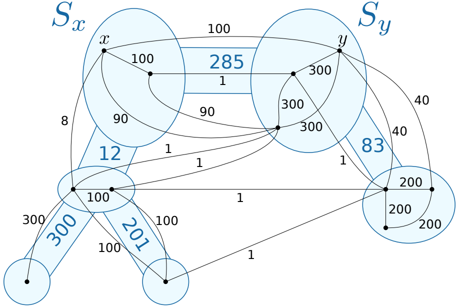


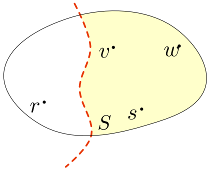




Gomory-Hu trees: lemma about cuts

Let S be minimal $r - s$ cut, and $s, v, w \in S$.

There is a minimal $v - w$ cut $T \subset S$.



Gomory-Hu trees: lemma about cuts

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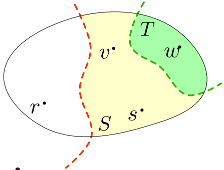
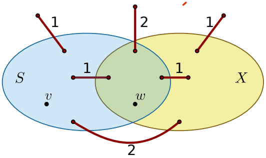


any $v - w$ cut X, s.t. $S \cap X \neq \emptyset$

$$A := \partial(S) + \partial(X)$$



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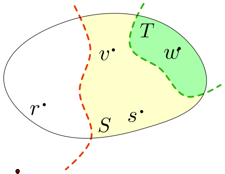

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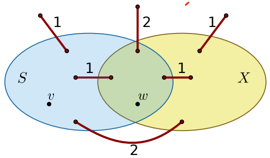
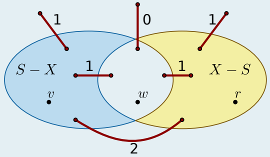


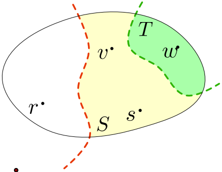
case 1: $r \in X$

$$B := \partial(S - X) + \partial(X - S)$$
$$\partial(S - X) + \partial(X - S) \leq \partial(S) + \partial(X)$$
$$\partial(X - S) \geq \partial(S)$$
$$\partial(S - X) \leq \partial(X) \Rightarrow S - X \text{ is min. } v - w \text{ cut}$$



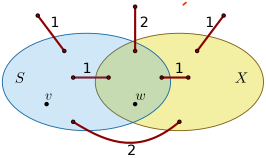
Gomory-Hu trees: lemma about cuts

Let S be minimal $r - s$ cut, and $s, v, w \in S$.
There is a minimal $v - w$ cut $T \subset S$.



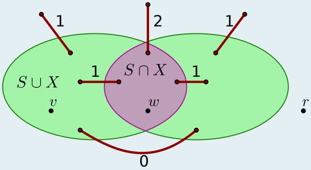
any $v - w$ cut X, s.t. $S \cap X \neq \emptyset$

$$A := \partial(S) + \partial(X)$$



case 2: $r \notin X$

$$B := \partial(S \cup X) + \partial(S \cap X)$$
$$\partial(S \cup X) + \partial(S \cap X) \leq \partial(S) + \partial(X)$$
$$\partial(S \cup X) \geq \partial(S)$$
$$\partial(S \cap X) \leq \partial(X) \Rightarrow S \cap X \text{ is min. } v - w \text{ cut}$$



Gomory-Hu trees

lemma

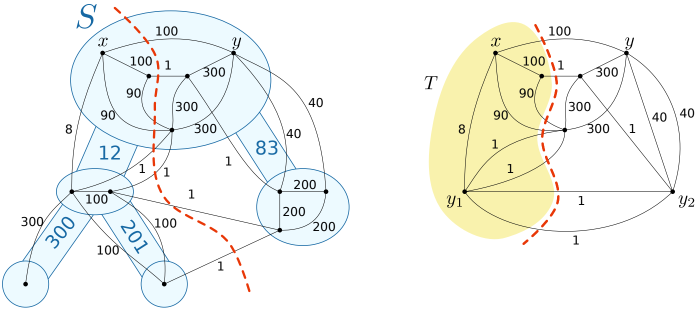
Let S be minimal $r-s$ cut, and $s, v, w \in S$. There is a minimal $v-w$ cut $T \subset S$.

Invariant

$\forall\ e = \left(S_i^{(t)}, S_J^{(t)}\right)\ \exists$ 2 witnesses $x \in S_i^{(t)}$, $y \in S_j^{(t)}$:

- $\omega'(e) = f_G(x,y)$
- M is minimal $x-y$ cut in G.

lemma $\Rightarrow$ the cut does not separate subtrees $\Rightarrow$ is min. cut in G



Gomory-Hu trees

lemma

Let S be minimal $r - s$ cut, and $s, v, w \in S$. There is a minimal $v - w$ cut $T \subset S$.

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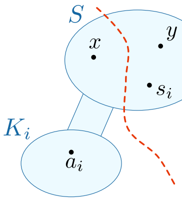


$f_G(a_i, x) = f_G(a_i, s_i)$

- $f_G(a_i, x) \leq f_G(a_i, s_i)$
- contract $S_y \Rightarrow \hat{G}$

$f_G(a_i, x) = f_{\hat{G}}(a_i, x)$

$f_{\hat{G}}(a_i, x) \geq \min\{f_{\hat{G}}(x, \hat{y}), f_{\hat{G}}(a_i, \hat{y})\}$

$f_{\hat{G}}(a_i, \hat{y}) \geq f_G(a_i, s_i)$, $f_{\hat{G}}(x, \hat{y}) \geq f_G(x, y)$

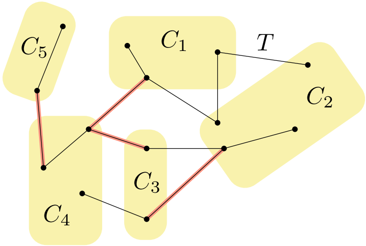
$f_G(x, y) \geq f_G(a_i, s_i)$

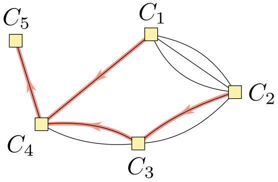
second variation: Min-k-Cut

Given are $G = (V, E)$, $\omega : E \mapsto \mathbb{R}^+$, and a number k.

Find a minimal cut that results in $\geq k$ components.

Algorithm: union of $k-1$ cheapest edges of Gomory-Hu tree





$$ALG \leq \sum_{i=1}^{k-1} \omega'(e'_i)$$

$$2 \cdot OPT = \sum_{i=1}^{k} \partial_G(C_i)$$

there are $k-1$ edges from GHT, such that

$$\omega'(e'_{h_i}) \leq \partial_G(C_i)$$